GENERAL MECHANICS

Being Volume I of
"INTRODUCTION TO THEORETICAL PHYSICS"

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THE MACMILLAN COMPANY 1949

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

A LARGE number of books on Mechanics have appeared more or less recently—some of them excellent—so that an author who ventures to add to their number feels bound to offer some explanation. During my long activity as a teacher I have frequently observed that the difficulties with which the student has to contend when he first enters the realm of theoretical physics are more often concerned not with the mathematical form, but with the physical content of the ideas which are presented to him. It is not the calculations with equations that cause him most trouble, but the setting up of the equations and, in particular, their interpretation. The chief purpose of the present volume is to lend him a helping hand in this respect. It is intended primarily for those students of science who are already in possession of a certain amount of mathematical knowledge, being familiar with the elements of Analytical Geometry and of the Infinitesimal Calculus. The particular method which I have proposed to myself is that of presenting the structure of Mechanics not as something already given, but as something which has been evolved step by step; the student is not, so to speak, led along in the direction traditionally prescribed by the classical writings of science, but rather is advised and occasionally warned at the decisive turning-points, in order that something of that particular pleasure may be retained which every person of independent thought experiences when advancing for the first time into a new field of science.

The fact that the manner of treating the subject-matter follows in general the same lines as were actually pursued when the science was being evolved, will be apparent to anyone who, like myself, inclines to the opinion that the history of an exact science does not deviate markedly from its structure as developed logically; this is true, of course, only as a whole, for often external circumstances, particularly such as are rooted in the idiosyncrasies of the pioneering investigator, have led to detours and even false routes, to follow which once again would be unnecessary and harmful to our purpose. Nevertheless, in deriving the theorems, I have by no means always sought the shortest and most elegant proof, but always that which has seemed to me to be the most suggestive and the most lucid. For my endeavour has been, not to represent either how the theorem was actually discovered or how it was subsequently proved most directly, but rather how it *could* have been found most simply. It must be conceded that this leaves a certain amount of freedom of play for the personal view.

There is not, of course, the slightest intention of treating any part of the subject completely, since the character of the book is elementary—as is indicated by its title. For an exhaustive treatment the reader is referred to the more comprehensive text-books on Mechanics and to the detailed special literature. Often, however, a fitting occasion arises for again proving in a new way a theorem that has already been derived earlier. For there is no better way of exhibiting in its true light the particular nature of a problem and also the power of the individual methods used to solve it than to treat one definite problem in different ways.

An alphabetical list of all the definitions used and of the most important theorems will, it is hoped, increase the usefulness of the book.

MAX PLANCK.

Berlin-Grunewald,
August, 1916.

PREFACE TO THE SECOND EDITION

THE printing of a new edition enables me not only to make some necessary corrections, but also to insert a few additions, some small and some considerable. Among the latter I must mention in particular the introduction of the partial differential equation of Hamilton and Jacobi which has recently become of paramount importance for the quantum theory. I wish to take this opportunity of thanking again for their kind interest those of my colleagues who suggested these changes.

MAX PLANCK.

Berlin-Grunewald,
December, 1919.

PREFACE TO THE FOURTH EDITION

THE character of the book has been preserved in this new edition. In particular I have continued to endeavour, in deriving each law, to use not the way which is the shortest formally, but that which follows the physical ideas most closely, and which approaches most nearly to that used originally. For the compactness of a formula often makes the relationship which it expresses appear simpler than it is in reality; this is because the real difficulty has been transferred to the definitions. As a first introduction to a branch of knowledge, it is essential, in my opinion, that the ultimate definitions should not be placed at the beginning as ready products, but that their usefulness and necessity must impress themselves only in the course of presentation in discussing definite problems.

Among the slight improvements that have been made I need mention only that I have here used for Lagrange's function (the kinetic potential), instead of the symbol H, used by Helmholtz, the now more generally used symbol L, and I have reserved the symbol H for the Hamiltonian function. But I could not persuade myself to represent the kinetic energy, which, following Boltzmann, I have hitherto denoted by L (vis viva), by means of the letter T, which is frequently used for it nowadays. For the letter T must be reserved for temperature, which often appears conjointly with kinetic energy, as in statistical thermodynamics. I have preferred to use for the kinetic energy the symbol K, which immediately suggests itself and which is hardly ever likely to give rise to confusion.

MAX PLANCK.

Berlin-Grunewald,
March, 1928.

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INTRODUCTION

§ 1. MECHANICS is the study of the laws of motion of material bodies. Motion is change of position in time. But the concept of motion involves not only the concepts of space and time but also the concept of what moves, and this need not in general be a material body. For example, we also speak of the motion of a crest of a wave on a water surface, which is of course to be carefully distinguished from the motion of the water particles themselves; or we speak of the motion of a shadow over a bright surface, or of the motion of a line of force in a magnetic field. In these cases what moves is not matter but a certain "state" on which we have fixed our attention. To characterize the motions of material bodies more clearly we therefore also call them "corpuscular" or "convective" motions. Mechanics is concerned only with these corpuscular motions, but this does not exclude the possibility that a corpuscular motion may simultaneously be regarded as a wave-motion, as instanced above in the case of the water wave. The view that all physical changes, and they include all kinds of motions, may be traced back to corpuscular motions is called the mechanical view of nature. We do not propose to discuss the question as to whether it is justified.

§ 2. The simplest material body is a material point, that is, a body whose spatial dimensions are vanishingly small compared with all the dimensions that play a part in its motion. The question whether a definite material body may be taken as a point thus depends on the nature of the motion under consideration. We may, for example, regard the earth in its motion around the sun as a material point, but not when dealing with its rotation about its axis; in fact, we can never regard a body which rotates about an axis which lies in its interior as a material point so far as this rotation is concerned.

The material point must be carefully distinguished from the geometrical point. The latter is completely characterized by the point at which it is situated, but the former also depends on the constitution of its matter; indeed, the material points are to be regarded from the very outset as different not only quantitatively but also qualitatively. For it is not possible to specify *ab initio* a general measure for the quantity of matter. For example, quantitative comparisons in the case of two different substances, say iron and lead, can be made only with reference to some special property.

A material body may always be regarded as composed of such small parts that each of them may be conceived as a material point, and, correspondingly, any motion of a body, no matter how complicated it may be, can be traced back to the motions of the material points of which it is composed. Hence we first consider a single material point. We therefore divide mechanics into two parts: the mechanics of a material point and the mechanics of a system of material points.

PART ONE

MECHANICS OF A MATERIAL POINT

CHAPTER I

MOTION ALONG A STRAIGHT LINE

§ 3. WE shall first consider the rectilinear motion of a material point in itself, such as it presents itself to direct observation, without inquiring into its causes (study of pure motion; kinematics, or phoronomy). A moving point changes its position with time; its motion is determined if we know its position at every arbitrary moment of time—that is, if its position is given as a function of the time. Its position is characterized by a geometrical point P, and the latter point is given by its distance x from a point O assumed fixed in space—namely, the origin of co-ordinates (Fig. 1). We assume the quantity x, the abscissa of the point P, to be positive or negative according as P lies to the right or to the left of O. Then P coincides with O when $x = 0$. The path of the point P is the x-axis, or the axis of abscissæ. The direction in which x increases is called the direction of the axis; it is denoted in Fig. 1 by an arrow. To be able to express the distance x by means of a definite number we must introduce a definite unit of length, for which we usually choose the centimetre; this is the hundredth part of the length of the standard metre which is preserved in Paris and which very nearly represents the ten-millionth part of the quadrant of the earth's meridian. The quantity x is then the number of centimetres which is found by measurement to be contained in the length OP.

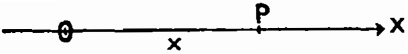

FIG. 1.

In precisely the same way as a definite position is indicated by a geometrical point P, so a definite time is characterized by a moment of time (or point in time)—namely, by the length of the time t which has elapsed since a point of time assumed definitely fixed, this initial point of time being measured by any clock that goes with sufficient regularity. We assume the co-ordinate of time t to be positive or negative according as the point of time is later or earlier than the initial moment, for which $t = 0$. As the direction of the time-axis we take the direction from earlier to later times. As a rule we shall take the second as the unit of time; this is the 86,400th part of the mean solar day. Then the quantity t denotes the number of seconds which have elapsed since the time $t = 0$.

The motion of the material point is determined when its position is given as a function of the time—that is, when:

$$x = f(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where we assume the function f to be real, one-valued, continuous and differentiable. For the material point occupies a definite position at any moment of time and does not leap suddenly to a new position.

If we solve equation (1) in terms of t, we get:

$$t = \phi(x)$$

which gives us the answer to the question as to when the material point is to be found at a definite point x. The function ϕ need not be either real or one-valued; for it may happen that the material point never actually reaches a definite point x, or, again, that it reaches the point a number of different times, as, for example, when the motion is periodic.

§ 4. As an illustration we shall first take the special case where the function $f(t)$ is linear—that is:

$$x = at + b, \quad . \quad . \quad . \quad (2)$$

where a and b are constants.

The physical meaning of the constant b is simple: it denotes the position of the point when $t = 0$. The meaning of the constant a becomes clear from the following reflection. Let us inquire into the length of the path which the point traverses in any interval of time $t' - t = \Delta t$. This comes out as $x' - x$, if:

$$x' = at' + b$$

that is:

$$x' - x = \Delta x = a(t' - t) = a \, . \, \Delta t.$$

Hence in the motion (2) that we have assumed, every distance Δx that has been traversed is proportional to the times Δt required to traverse it, or equal distances are covered in equal times. We see then that the quantity a is precisely the constant ratio of a distance to the time required to traverse it—namely:

$$\frac{\Delta x}{\Delta t} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and this ratio is called the *velocity* of the moving point. It is the path traversed in unit time; the path is reckoned positive or negative according as x increases or decreases when t increases. The motion (2) here considered, in which the velocity is constant, is therefore called a "uniform" motion.

Let us now take the general case of any arbitrary motion: $x = f(t)$, and let us again inquire what distance is traversed by the moving point in any interval of time $t' - t = \Delta t$. This again comes out in an analogous way as $x' - x$, if $x' = f(t')$. Thus:

$$x' - x = \Delta x = f(t') - f(t) = f(t + \Delta t) - f(t).$$

Division gives:

$$\frac{\Delta x}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

This ratio of a distance to the time required to traverse it is called the *mean* (or *average*) *velocity* of the moving point during the interval of time between t and $t + \Delta t$. Thus in general the mean velocity depends on both t

If we make the interval of time smaller and smaller we ultimately get the limiting value:

$$\lim. \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x} = \dot{f}(t) \quad . \quad . \quad . \quad . \quad (4)$$

and we call this differential coefficient the *velocity* u of the moving point at the time t. This quantity now depends only on the time t itself.

For a point which is in uniform motion we again get from (2) for the velocity: $u = \frac{dx}{dt} = a$; for a point which is at rest $x = \text{const.}$, $u = 0$.

§ 5. Before we can express the value of a velocity by means of a definite number, the units of length and time must of course first be fixed. According to the choice of these units the physical meaning of a number which serves to represent a velocity varies. Hence we say that velocity is not a "pure" number, but has a "dimension"—namely, the dimension of a length divided by a time:

$$\left[\frac{l}{t}\right].$$

This symbol, which was introduced by Maxwell to express dimensions, at the same time indicates how the numerical value which is to be written for a definite velocity alters when the length or the time or both are to be altered. For example, if we wish to refer the velocity:

$$20\left[\frac{\text{cm.}}{\text{sec.}}\right]$$

to metres and minutes, we have only to write:

$$1\,[\text{cm.}] = \frac{1}{100}\,[\text{metre}], \quad 1\,[\text{sec.}] = \frac{1}{60}\,[\text{minute}],$$

and we can now calculate with these symbols as with mathematical quantities. Substitution then leads to the required result:

$$20\left[\frac{\text{cm.}}{\text{sec.}}\right] = 12\left[\frac{\text{metre}}{\text{minute}}\right].$$

We can proceed in the same way with any derived quantity so soon as its dimensional formula is known.

§ 6. After the case of uniform motion u = const. we now consider the special case where the velocity depends linearly on the time, thus:

$$u = a_1 t + b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where a_1 and b_1 are constants. The constant b_1 is called the velocity of the point for $t = 0$. The significance of the constant a_1 emerges from the following consideration.

Let us inquire into the change which the velocity experiences during any interval of time $t' - t = \Delta t$. This is equal to $u' - u$ if:

$$u' = a_1 t' + b_1.$$

Thus:

$$u' - u = \Delta u = a_1(t' - t) = a_1 . \Delta t.$$

In the motion that we have assumed the velocity thus always changes proportionally to the time, and the constant ratio of the change of velocity to the time in which the change occurs is the quantity a_1:

$$\frac{\Delta u}{\Delta t} = a_1, \quad . \quad . \quad (6)$$

and is called the *acceleration* of the moving point. It is the increase of velocity per unit of time and is positive or negative according as the velocity u increases or decreases as t increases. Hence the motion (5) here considered, in which the acceleration is constant, is also called a "uniform acceleration."

Let us now consider the general case of any arbitrary motion, that is, by (4):

$$u = f(t),$$

and let us again inquire into the change of velocity Δu during any interval of time $t' - t = \Delta t$. In a manner analogous to that given above this is again equal to

$u' - u$ if $u' = \dot{f}(t')$, and so, if we divide by $t' - t$, we get:

$$\frac{u' - u}{t' - t} = \frac{\Delta u}{\Delta t} = \frac{\dot{f}(t + \Delta t) - \dot{f}(t)}{\Delta t}.$$

This ratio of a change of velocity to the time during which it occurs is called the *mean acceleration* of the moving point in the interval of time between t and $t + \Delta t$. Hence the mean acceleration depends in general on both t and Δt.

If we now make the interval of time smaller and smaller, we finally get the limiting value:

$$\text{lim.}\,\frac{\Delta u}{\Delta t} = \frac{du}{dt} = \dot{u} = \ddot{x} = \dot{f}(t) \quad . \quad . \quad . \quad (7)$$

and this differential coefficient is called the *acceleration* of the moving point at the time t. It now depends only on the time t itself.

For a point which is moving uniformly as well as for one which is at rest the acceleration $\dot{u} = 0$.

The dimensions of an acceleration are, as we see from (7):

$$\left[\frac{l}{t^2}\right].$$

Hence, for example (cf. § 5), the acceleration:

$$20\left[\frac{\text{cm.}}{\text{sec.}^2}\right] = 720\left[\frac{\text{metres}}{\text{min.}^2}\right].$$

We may, of course, pursue this method still further and define "accelerations of a higher order." But such quantities play only a small part in physics.

If one of the quantities x, u, $\dot{u}$ is given as a function of the time t the other two can be found by differentiation or integration with respect to t. For example, in the case of the motion of uniform acceleration (5) the co-ordinate x depends quadratically on t.

§ 7. So far we have spoken only of the motion itself without considering its causes. We shall now also take the latter into consideration and for this purpose we must refer back to particular observations. Experience has made us familiar with motions of very different kinds—for example, those of a thrown ball, a falling stone, a vibrating pendulum. In every case we note that a definite cause can be given for the type of motion: in the case of the thrown ball the cause is given, say, by the taut muscles of our arms; in that of the falling stone it is the earth; in that of the vibrating pendulum it is, besides, the method of suspension. This is only to express that if the bodies mentioned (arm, earth, method of suspension) were not present, the motion in question would not occur in the manner observed. The main object of mechanics is to find the motion which results from a prescribed cause.

The first question which we shall answer is this: How does a material point move if we disregard its previous history and if we eliminate all the causes which may previously have influenced its motion—that is, if the material point is now completely isolated in empty space at an infinite distance from all other bodies? Of course, this experiment cannot be carried out exactly in practice; indeed, it may be doubted whether this question has a physical meaning at all. For it can never be determined with certainty whether there are not enormously great bodies at enormously great distances which have an appreciable influence on the motion of the point. On the other hand, in the case of any special motion we can reduce the influence of the bodies which, as we know, come into question as causes of the motion, and, in principle, we can reduce this influence to an unlimited extent. Thus we can allow the thrown ball to follow its course freely, we can cut through the pendulum thread, and so forth. We cannot, of course, remove the earth, but we can eliminate its influence by making the material point move on a fixed plane which is accurately horizontal—for example, on the surface of a suitably large billiard table. Experiment then shows that the material point—for example, a billiard ball

—will move in a straight line with gradually decreasing velocity. But the decrease of velocity occurs the more slowly the more the plane base is free from unevenness; on a very smooth ice-surface the decrease of velocity is very much less than on the billiard cloth. From this it is concluded that on an absolutely plane surface, in which surface phenomena due to rubbing and to warming resulting from roughnesses are excluded, the decrease of velocity would be zero—that is, the velocity would be constant. Hence we answer the above question by saying that *a material point which is deprived of all causes of motion moves uniformly and rectilinearly*, in accordance with equation (2). (Principle of Inertia, Newton's First Law of Motion.)

The above derivation is by no means intended as a proof of the law of inertia. It merely serves to describe a way by which we can arrive at an enunciation of the principle. The proof of the law is to be sought only in the confirmations which its innumerable applications have presented. Its significance consists actually in the fact that it expresses in a single sentence the sum of all the observations that have been collected in this field.

On the other hand, we must not regard the principle of inertia as obvious or as a mere definition; for it contains a definite physical statement the correctness of which can be tested by experiment to a high degree of accuracy.

§ 8. Let us now take the case where a material point which was originally completely isolated—that is, was moving uniformly and rectilinearly, say a sphere on an absolutely smooth horizontal plane—is accelerated or retarded by some cause of motion in the direction of its motion. If we produce a change of motion by means of our muscles by pushing the sphere from behind, in the case of positive acceleration, or obstructing it in front, in the case of negative acceleration, then we experience a feeling of exertion which is incapable of being defined more precisely as a sensation, but whose intensity is certainly related causally to the amount of acceleration produced.

We shall therefore use the sensation of our muscular sense as a measure for the cause of the acceleration and so call the cause of the acceleration the "force" X, which we exert on the sphere. Experiment then teaches us that a more intense muscular sensation—that is, a greater force X—corresponds to a greater acceleration $\dot{u}$ and that the direction of the acceleration is reversed when the direction of the force is reversed. When $X = 0$, then $\dot{u} = 0$ in accordance with the principle of inertia.

We can get no further than this by experimental means in determining the relationship between force and acceleration, because our muscular sensations are far too indefinite and fluctuating to give us an exact measure of the value of the force that is exerted. We bridge over this gap by proposing a more precise definition. *We set the force X proportional in magnitude and sign to the acceleration $\dot{u}$ that is produced.* (Newton's Second Law of Motion.) We may do so because this new convention, so far as an experimental test is possible at all, agrees with the relationship between X and $\dot{u}$ already fixed above, which was derived from our muscular sensations. Moreover, it has the advantage, which we shall presently make use of, that we can immediately apply it to the general case where the acceleration is not produced by our muscles at all but by any other body, so that there can be no question of a sense-impression. So we now define quite generally for any arbitrary motion that the cause of the motion is a force and we set its value proportional to the acceleration produced by it. This value corresponds to the exertion which we should experience if we were to produce this change of motion with our muscles instead of with the causative bodies.

The question immediately suggests itself to us whether it would not be simpler, and hence more rational, to define force from the very outset by means of acceleration and not to proceed indirectly by means of muscular sensation. To this it must, however, be objected that the concept of force is something quite different from that of acceleration and that we get much nearer to the content of this concept by bringing it into relationship with the muscular sense than with acceleration. This will manifest itself clearly in the next section and also repeatedly later on, for example, in the case of relative motions (§ 57).

Moreover, this method of defining a fundamental physical concept by first referring it back to a specific sense-impression and then supplementing and refining this first primitive definition by means of a second definition is the one usually adopted in physics and is probably the only possible one. For example, we first define the degree of warmth of a body by means of the heat sense and the colour of a ray of light by our colour sense. For exact use, however, these definitions must be refined, and this is done in every case by referring them to a phenomenon which is susceptible of accurate measurement: in the case of heat reference is made to volume changes (thermometer), in the case of colour to wave-length (interference fringes). If we wished to define heat directly by means of volume changes or colour directly by means of wave-length, just as we measure force directly by means of acceleration, then these concepts would lose just that significance which has made them of value for more exact investigation and which has smoothed the way for the further development of physical theories (a matter of still greater importance).

Actually, the definition of force which is based on acceleration is not the final definition, but is capable of being further improved and generalized, as will be shown later (§ 124).

§ 9. It would obviously be simplest to set the force X not only proportional to the acceleration $\dot{u}$, but directly equal to it. But this would bring us into conflict with the primary definition of force based on our muscular sense, for then a definite force would have to produce a definite acceleration under all circumstances. Let us take two spheres, one of wood, the other of iron, say of the same size, both moving with the same constant velocity on a smooth horizontal plane. Experiment then teaches us that it requires a greater effort to accelerate or retard the iron sphere in a definite way than the wooden sphere. Hence we say that the iron sphere " has more inertia " than the wooden sphere and we have to insert a (positive) constant of proportionality in the relationship between force and acceleration :

$$X = m\dot{u} = m\frac{du}{dt} = m\frac{d^2x}{dt^2} \quad . \quad . \quad (8)$$

which is determined by the *constitution* of the moving material point (§ 2). Since a greater force is necessary to produce a definite acceleration in the case of the sphere which has greater inertia, m is greater in this case; hence in general we call m the *inertial mass* of the material point. This is of course the same for all the different kinds of motion of the point and for all the different forces which act on it.

We take as the unit of mass m the mass of a perfectly definite part of a certain body—namely, the 1000th part of the mass of the standard piece of platinum preserved in Paris—and we call this part 1 gramme. It is very nearly equal to the mass of 1 c.c. of water at 4° C.

By fixing the unit of mass we of course also, by (8), fix the unit of force. Moreover, a force has the dimensions :

$$\left[\frac{ml}{t^2}\right] \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

§ 10. As a first application of the fundamental equation (8) we discuss the motion of a material point which is projected vertically upwards in a vacuum. After the point has been projected and is left to itself, only the attractive force of the earth acts on it : we call this force the " weight " G of the point and regard it as constant, acting in a direction vertically downwards. If we choose the positive x-axis in the upward direction, then :

$$X = -G \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituted in (8) this gives:

$$m\frac{du}{dt} = -G.$$

Integrating, we get:

$$mu = -Gt + C.$$

The constant of integration C can be calculated if the velocity u is known for a definite moment of time t, for example, for the initial moment $t = 0$. If we call the (positive) initial velocity u_0, then when $t = 0$ and $u = u_0$, we get from the last equation that:

$$mu_0 = C.$$

Hence by substitution:

$$mu = -Gt + mu_0,$$

or:

$$u = -\frac{G}{m}t + u_0 = \frac{dx}{dt} \quad . \quad . \quad . \quad . \quad (10)$$

Thus the velocity u decreases uniformly as the time t increases. For $t = \frac{mu_0}{G}$ it becomes equal to zero, and after that becomes negative—that is, the material point begins to fall. By integrating once again we get from (10):

$$x = -\frac{1}{2}\frac{G}{m}t^2 + u_0t + C',$$

and if $x = x_0$ when $t = 0$, then:

$$x = x_0 + u_0t - \frac{1}{2}\frac{G}{m}t^2 \quad . \quad . \quad . \quad . \quad (11)$$

This fully describes the motion.

The maximum height x_m attained—that is, the maximum value of x—is obtained if we substitute in (11) the value of t corresponding to the moment in which the velocity is reversed:

$$x_m = \frac{1}{2}\frac{m}{G}{u_0}^2 + x_0 \quad . \quad . \quad . \quad . \quad (12)$$

By eliminating t from (10) and (11) we get the answer to the question as to what velocity u the point has at a definite place x:

$$u^2 - u_0^2 = \frac{2G}{m}(x_0 - x) \quad . \quad . \quad . \quad (13)$$

For $x > x_m$, u becomes imaginary, as is natural; for $x = x_m$, u becomes equal to zero, and for $x < x_m$, u has two equal and opposite values, the positive value corresponding to ascent, the negative to descent. Thus the descending motion is fully symmetrical with the ascending motion.

If x and u_0 are not known, the integration constants C and C' remain indeterminate in the equations of motion. Hence the two quantities which denote the initial position and the initial velocity of the material point are summarized in the term "initial state," and we may enunciate the theorem that if the acting force and the initial state are given the motion is determined in all its details. In general, we take the " state " of a material point to denote comprehensively its position and its velocity.

The preceding laws which govern the motion of falling material points were first established experimentally by Galileo. He also found that the quotient $\frac{G}{m}$ is the same for all material points; that is, if we set:

$$\frac{G}{m} = g \quad . \quad . \quad . \quad . \quad (14)$$

then the quantity g, the acceleration due to gravity, does not depend on m. On the other hand, g is slightly different at different places. Its value increases as we pass from the equator of the earth to the poles—namely, from $978 \left[\frac{\text{cm.}}{\text{sec.}^2}\right]$ to $983{\cdot}2 \left[\frac{\text{cm.}}{\text{sec.}^2}\right]$.

Hence the weight $G = mg$ of a definite material point is different at different places on the earth. The weight of 1 gramme amounts to 978 dynes at the equator and 983·2 dynes at a pole.

§ 11. The circumstance that the acceleration g due to gravity of a material point is independent of its mass gives us a very exact method of measuring masses.

Suppose we have two exactly equal vessels fastened at the ends of a string which runs over a fixed pulley, and that a material point of mass m is placed in the one vessel and a certain quantity of water of mass m' is placed in the other. Then the pulley will begin to turn in the direction in which the string is being pulled the more strongly—that is, in which, by § 8, the greater force is acting. The string will therefore remain at rest permanently if the two forces are equal—that is, if the weight G of the material point is equal to the weight G' of the water that has been poured into the second vessel, or by (14), if:

$$m = m'.$$

Now by § 9 m' is equal to the volume of the water, expressed in cubic centimetres. So we get the law: the mass of a material point is equal to the volume of water which keeps it in equilibrium. Hence we are not concerned at all about the value of the acceleration g due to gravity: a material point weighs the same number of grammes everywhere, since the weight G of the point changes from place to place in the same ratio as the weight G' of the corresponding volume of water. To prove that G changes we could, for example, use an elastically extensible string in the above experiment. Then the two halves of the string would be stretched more by the same bodies and hence would be longer at the north pole of the earth than at the equator.

§ 12. Particular interest attaches in physics to those forces which express themselves as attractions and repulsions and whose value depends only on the distance which separates the points between which they act; these are the so-called "central forces."

Let us discuss the case of the rectilinear motion of a material point which is attracted to a fixed centre with a force which is proportional to its distance from the centre.

If we take the centre as our origin of co-ordinates, then the distance of the moving point P from the centre is equal to x, and the attractive force is, in magnitude and direction :

$$X = -cx\,(c > 0).$$

Hence we get the equation of motion (8) :

$$m\frac{du}{dt} = -cx \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the initial state, $t = 0$, let :

$$x = 0 \text{ and } u = u_0\,(> 0) \quad . \quad . \quad . \quad (16)$$

To integrate the equation of motion we multiply both sides by $\frac{dx}{dt} = u$ and obtain :

$$m\,.\,u\,.\frac{du}{dt} = -cx\frac{dx}{dt},$$

or, integrating with respect to t :

$$\frac{1}{2}mu^2 = -\frac{c}{2}x^2 + C,$$

and, since when $x = 0$, $u = u_0$, we get :

$$mu^2 = mu_0^2 - cx^2 = m\left(\frac{dx}{dt}\right)^2 \quad . \quad . \quad . \quad (17)$$

From this we see among other things that the velocity u never exceeds u_0 and that the distance x never exceeds $u_0\sqrt{\frac{m}{c}}$.

To perform the second integration we write the last equation in the form :

$$dt = \frac{dx\,.\sqrt{m}}{\sqrt{mu_0^2 - cx^2}}.$$

Integration then gives :

$$t = \sqrt{\frac{m}{c}}\,.\,\sin^{-1}\left(\frac{x}{u_0}\sqrt{\frac{c}{m}}\right) + C'.$$

From the initial condition (16) we get $C' = 0$; and hence:

$$x = u_0 \sqrt{\frac{m}{c}} \sin\left(\sqrt{\frac{c}{m}} \cdot t\right) \quad . \quad (18)$$

Thus the motion is a periodic vibration with the fixed point of attraction as centre.

The constant factor in front of the sine is called the "amplitude," the angle under the sine, which varies with the time, is called the "phase," the constant factor preceding t is called the "angular or radian frequency" of the vibration (number of vibrations in the time 2π). The derivation of a period of vibration is $2\pi\sqrt{\frac{m}{c}}$ and hence, like the frequency, does not depend on the initial velocity u_0, nor on the initial position; since the case of any arbitrary initial position x_0 can be directly reduced to the case here discussed by transferring the initial point of the time t to the moment where $x = 0$.

§ 13. The special law of motion here discovered plays a very important part in physics, for it holds quite generally for small oscillations of a point about a stable position of equilibrium, as can easily be proved.

If a point which is originally at rest is disturbed from its position of equilibrium by a blow which imparts to it an initial velocity u_0, then if its position of equilibrium was stable a force acts on the point at every moment, which pulls it back to its position of equilibrium and which we suppose to depend on its position in some way, thus:

$$X = f(x).$$

Let $x = 0$ denote the position of equilibrium.

If the oscillations are sufficiently small, we can expand $f(x)$ in a power series:

$$X = c_0 + c_1 x + c_2 x^2 + \ . \ . \ .,$$

in which the first constant $c_0 = 0$, since $X = 0$ for $x = 0$, and the second constant c_1 is negative, since the equilibrium is to be stable. If the terms of the series which are of a smaller order of magnitude are omitted we obtain precisely the motion treated in the preceding section. So we get the general theorem that the period of a small rectilinear vibration about a stable position of equilibrium is independent of the nature of the disturbance. It will be found later, in § 70, that the same law also holds for non-rectilinear oscillations.

§ 14. If several forces act on a material point simultaneously in the same or in opposite directions, which we shall express by X_1, X_2, X_3, . . ., these forces are equivalent to a single force X which can be represented in magnitude and direction by:

$$X = X_1 + X_2 + X_3 + \ldots \qquad (19)$$

We say that the individual forces combine to form the "resultant" force X. If $X = 0$ the individual forces are in equilibrium and the material point behaves in every way as if no force acted on it at all.

§ 15. As an example we consider the case of the rectilinear motion of a material point which, as in § 12, is attracted to the origin of co-ordinates by the force cx, but is at the same time "damped" in its motion by friction or some other cause owing to the action of a force whose magnitude is proportional to its instantaneous velocity u. Then by (19) the resultant force is:

$$X = X_1 + X_2$$

where:

$$X_1 = -cx \text{ and } X_2 = -\rho\frac{dx}{dt},$$

(ρ is a constant coefficient of friction)

and the equation of motion (8) runs:

$$m\frac{d^2x}{dt^2} = -cx - \rho\frac{dx}{dt} \qquad (19a)$$

or, if we use the abbreviations:

$$\frac{c}{m} = a \text{ and } \frac{\rho}{2m} = w,$$

then:

$$\frac{d^2x}{dt^2} + 2w\frac{dx}{dt} + ax = 0 \quad . \quad . \quad . \quad . \qquad (20)$$

As in § 12 let the initial state again be given by:

$$x = 0,\ u = u_0\,(> 0).$$

A particular integral of the differential equation (20) is:

$$x = Ae^{at},$$

where the constant A is arbitrary, but the constant α must satisfy the equation:

$$\alpha^2 + 2w\alpha + a = 0$$

Let us call the two roots of this quadratic equation α and β, then:

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = -w \pm \sqrt{w^2 - a} \quad . \quad . \quad . \qquad (21)$$

and the expression:

$$x = Ae^{at} + Be^{\beta t} \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

is also an integral of equation (20); it is, in fact, the general integral, since it contains two arbitrary constants A and B.

From (22) we get by differentiating:

$$\frac{dx}{dt} = u = A\alpha e^{at} + B\beta e^{\beta t} \quad . \quad . \quad . \quad . \qquad (23)$$

The values of the constants of integration A and B are determined by the initial state. For if $t = 0$ it follows from (22) and (23) that:

$$0 = A + B \text{ and } u_0 = A\alpha + B\beta$$

Consequently if we calculate the values of A and B and substitute in (22) and (23), then:

$$x = \frac{u_0}{\alpha - \beta}(e^{at} - e^{\beta t}) \quad . \quad . \quad . \qquad (24)$$

$$u = \frac{u_0}{\alpha - \beta}(\alpha e^{at} - \beta e^{\beta t}) \quad . \quad . \quad . \quad . \qquad (25)$$

These equations, taken together with (21), completely determine the motion. To investigate its more detailed peculiarities we shall consider successively the cases where the square root in (21) is real, zero or imaginary.

1. Let $w^2 > \alpha$. Then α and β are both negative; and $-\beta > -\alpha$. Hence x comes out as positive for all times t except for $t = \infty$, when x becomes equal to zero. The motion is aperiodic; the moving point attains its position of greatest displacement—that is, the maximum value of x for $u = 0$ and:

$$t = \frac{\log \frac{\beta}{\alpha}}{\alpha - \beta},$$

and then returns immediately to its position of equilibrium.

2. Let $w^2 = \alpha$. Then, by (21):

$$\alpha = \beta = -w.$$

Since the expression for x in (24) assumes the form $\frac{0}{0}$ for this case, we get the true value by setting $w^2 - a = \epsilon^2$, so that:

$$\alpha = -w + \epsilon, \ \beta = -w - \epsilon.$$

We now insert these values in (24) and proceed to the limit obtaining. In this way:

$$x = u_0 t e^{-wt}, \ u = u_0 e^{-wt}(1 - wt) \quad . \quad . \quad (26)$$

The motion is again aperiodic, the displacement x is always positive; and its maximum value is $\frac{u_0}{ew}$, which it attains at the time $t = \frac{1}{w}$.

3. Let $w^2 < a$. Then, by (21), α and β are conjugate imaginaries, namely:

$$\left.\begin{matrix}\alpha \\ \beta\end{matrix}\right\} = -w \pm i\sqrt{a - w^2}, \text{ where } i = \sqrt{-1}$$

Substituting in (24) we get :

$$x = \frac{u_0}{\sqrt{a - w^2}} \cdot e^{-wt} \, . \sin(t \, . \sqrt{a - w^2}) \quad . \quad (27)$$

The material point executes damped oscillations and comes to rest at the time $t = \infty$. For $t = \frac{n \, . \, \pi}{\sqrt{a - w^2}}$ (where n is an arbitrary integer) the point passes through the position of equilibrium, in the positive direction when n is even, and in the negative direction when n is odd. The duration of a period is the time which elapses between two successive transitions in the same direction through the position of equilibrium—that is, $\frac{2\pi}{\sqrt{\alpha - w^2}}$; it increases as the resistance w increases, but, as in the case of undamped oscillations, it is independent of the initial state.

The velocity u comes out as :

$$u = u_0 e^{-wt} . \left\{\cos(t . \sqrt{a - w^2}) - \frac{w}{\sqrt{a - w^2}} \sin(t \, . \sqrt{a - w^2})\right\} \quad . \quad (28)$$

Hence for a transition through the equilibrium position in the positive direction we have :

$$u = u_0 e^{-\frac{2n\pi w}{\sqrt{a - w^2}}} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

These velocities increase in geometrical progression for the successive transitions ($n = 0, 1, 2, 3, \ldots$) or the natural logarithms of the velocities decrease in arithmetic progression—namely, by the amount $\frac{2\pi w}{\sqrt{a - w^2}}$ in each transition. This number is therefore called the "logarithmic decrement" of the oscillations, and since it is constant these oscillations are said to be "uniformly damped."

The amplitudes of the oscillations—that is, the maximum displacements—do not result from (27), say, by setting the sine equal to 1, but from (28) by putting $u = 0$ in it.

They have the same logarithmic decrement as the velocities (29) in passing through the position of equilibrium.

For $w = 0$ the oscillations become periodic and undamped, and the equations of motion become identical with those derived in § 12. For $w = \sqrt{a}$, (27) and (28) again reduce to the limiting case (26) which has already been discussed.

CHAPTER II

MOTION IN SPACE

§ 16. As in the case of rectilinear motion in § 3, so here we treat the motion of a material point at first entirely without regard to its causes—that is, purely as a problem of phoronomy. The motion of a point in space is determined when its position is given as a function of the time t. To characterize the position of a point in three-dimensional space three co-ordinate axes are necessary; we shall assume these axes to be mutually perpendicular and shall denote their positive directions by x, y, z.

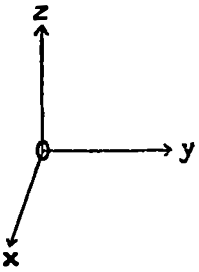

Fig. 2*a*

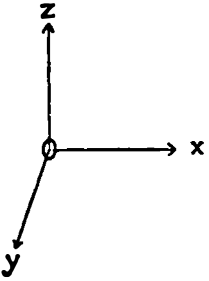

Fig. 2*b*.

This convention does not yet, however, determine the nature of the co-ordinate system; rather, an ambiguity remains which is illustrated in the two Figures 2*a* and 2*b*. It is clear that the two co-ordinate systems shown in these figures cannot be made to coincide exactly by displacing or rotating one or other in any way; they are related as the right hand to the left hand. But any other rectangular co-ordinate system can be made to coincide completely with either the system *a* or the system *b* by means of a displacement and a rotation.

Hence all co-ordinate systems fall into the two groups *a* and *b*, which may be characterized in the following way: if we open our hands so that the thumb, index-finger and middle-finger are mutually perpendicular and if we take the thumb as pointing in the x-direction, the index-finger in the y-direction and the middle-finger in the z-direction, then the right hand represents a co-ordinate system of the group *a* and the left hand a system of the group *b*. Hence *a*-systems are also called right-handed and *b*-systems left-handed. We shall always use right-handed systems here, as in Fig. 2*a*, unless the contrary is expressly specified.

§ 17. Instead of using the co-ordinates x, y, z, to characterize the position of a point P in space we often use its distance r from the origin O and the angles ξ, η, ζ which the direction from O to P makes with the positive co-ordinate axes. Then $OP = r$ is the diagonal of a rectangular parallelepiped the lengths of whose edges are x, y, z (Fig. 3) and we have:

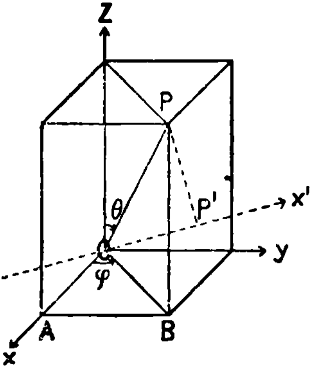

FIG. 3.

$$r^2 = x^2 + y^2 + z^2 \quad . \quad . \quad . \quad (30)$$

$$\cos \xi = \frac{x}{r}, \ \cos \eta = \frac{y}{r}, \ \cos \zeta = \frac{z}{r} \quad . \quad . \quad (31)$$

We always assume r to be positive and the direction-angles ξ, η, ζ to lie between 0 and π. Hence to a negative co-ordinate there always corresponds an obtuse direction-angle. The values of r, ξ, η, ζ are then uniquely determined by x, y, z, and *vice versa*. But the angles ξ, η, ζ cannot be chosen independently of one another; rather, according to the last two equations, they must satisfy the identity:

$$\cos^2 \xi + \cos^2 \eta + \cos^2 \zeta = 1 \quad . \quad . \quad . \quad (32)$$

Hence by (31):

$$\cos\xi : \cos\eta : \cos\zeta = x : y : z \quad . \quad . \quad . \quad (33)$$

These three cosines, the sum of whose squares = 1, are briefly called "direction-cosines" and their ratios "direction-ratios."

The distance *OP* measured in the direction (ξ, η, ζ), which uniquely determines the position of the point *P* in space is called a "directed magnitude" or a "vector," and, like all other vectors, is here denoted by the corresponding letter in clarendon type, $\boldsymbol{r}$. The numerical value of $\boldsymbol{r}$, the "absolute value" or the "magnitude" of the vector is:

$$r = |\boldsymbol{r}| \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

It is important to see clearly the distinction between r and $\boldsymbol{r}$. For example, if we have two points *P* and *P′*, the equation $r = r'$ denotes that *P* and *P′* are equally far from the origin *O*, but the equation $\boldsymbol{r} = \boldsymbol{r}'$ denotes that *P* and *P′* coincide, and the equation $\boldsymbol{r} = -\boldsymbol{r}'$ that *P* and *P′* lie at the same distance from *O* but on opposite sides.

The quantities x, y, z defined by (31) are called the "components" of the vector $\boldsymbol{r}$ in the direction of the co-ordinate axes. They are the projections of the distance *OP* on the co-ordinate axes.

In general the component x' of a vector $\boldsymbol{r}$ in any arbitrary direction is defined as the projection of the distance $|\boldsymbol{r}| = r$ in this direction—that is:

$$x' = r\cos\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where δ is the angle (acute or obtuse) which the direction of x' makes with the direction of $\boldsymbol{r}$.

If the direction angles ξ', η', ζ' of x' are given, the component x' (and the angle δ, which is not marked in Fig. 3) is calculated as follows. Instead of projecting the distance $r = OP$ directly on the x'-direction, we first project the distance $OA = x$ (Fig. 3), then the distance $AB = y$ and lastly the distance $BP = z$ all on the x'-direction.

That is, we allow a point to travel rectilinearly from O through A and B to P and at every moment drop a perpendicular on the x'-direction. The projection of the moving point—that is, the foot of the perpendicular—then traverses in its whole journey the rectilinear distance from O to P', the projection of P. Thus the algebraic sum of the three distances that are projected on x' is equal to the distance of the origin O from P'; accordingly:

$$x \,.\, \cos\xi' + y \,.\, \cos\eta' + z \,.\, \cos\zeta' = x' \quad . \quad . \quad (36)$$

Hence, by (31) and (35):

$$\cos\delta = \cos\xi\cos\xi' + \cos\eta\cos\eta' + \cos\zeta\cos\zeta' \quad . \quad (37)$$

By equation (35) the component of a vector $\boldsymbol{r}$ in its own direction ($\delta = 0$) is equal to r, that in the opposite direction ($\delta = \pi$) is equal to $-r$, and that in any perpendicular direction $\left(\delta = \frac{\pi}{2}\right)$ is equal to zero.

The equation (36) tells us that the component x' of a vector $\boldsymbol{r}$ in any direction (ξ', η', ζ') may also be obtained by starting out, not from the absolute value of the vector $\boldsymbol{r}$ but from its three components x, y, z and by forming from each of these components the component in the direction (ξ', η', ζ') and then algebraically adding the amounts so obtained. Hence in this respect, too, the three rectilinear components x, y, z are completely equivalent to the vector $\boldsymbol{r}$ itself.

§ 18. The motion of the point P in space is determined if its three co-ordinates x, y, z are given as functions of the time t:

$$x = f(t), \; y = \phi(t), \; z = \psi(t) \quad . \quad . \quad . \quad (38)$$

where the functions f, ϕ, ψ are assumed to be real, single-valued, continuous and differentiable. They, of course, also determine the orbit of the point; this is a certain curve in space, the two equations of which are obtained if the time t is eliminated from the three equations (38).

We now define, as in § 4, the three quantities:

$$\left.\begin{aligned}\frac{dx}{dt} &= \dot{x} = u\\ \frac{dy}{dt} &= \dot{y} = v\\ \frac{dz}{dt} &= \dot{z} = w\end{aligned}\right\} \quad . \quad . \quad . \quad (39)$$

and call them the "components of the velocity in the direction of the co-ordinate axes" of the point P at the time t. These are the velocities with which the projections of P on the co-ordinate axes move rectilinearly. Following on this we get a more general definition by differentiating (36) and so have as the velocity-component of P in any arbitrary direction (ξ', η', ζ') the velocity:

$$\frac{dx'}{dt} = \dot{x}' = u' = u\cos\xi' + v\cos\eta' + w\cos\zeta' \quad . \quad (40)$$

with which the projection P' of the point P moves rectilinearly in this direction.

On the basis of this definition we can prove that velocity is a vector. For if we set:

$$u^2 + v^2 + w^2 = q^2 \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\frac{u}{q} = \cos\lambda, \ \frac{v}{q} = \cos\mu, \ \frac{w}{q} = \cos\nu, \quad . \quad . \quad (42)$$

with the limitation that q must be positive and the direction-angles λ, μ, ν must lie between 0 and π, then, by (40):

$$u' = q(\cos\lambda\cos\xi' + \cos\mu\cos\eta' + \cos\nu\cos\zeta'),$$

and by (37):

$$u' = q\cos\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (42a)$$

where ϵ denotes the angle between the directions (ξ', η', ζ') and (λ, μ, ν). The component u' is thus the projection of the distance q, drawn in the direction λ, μ, ν, on the x'-direction.

This directed quantity is called the "velocity vector" and is denoted by the letter $\boldsymbol{q}$ in clarendon type:

$$\boldsymbol{q} = \dot{\boldsymbol{r}} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Hence the differentiation of a vector $\boldsymbol{r}$ with respect to the time does not mean the differentiation of its absolute value r, but rather it denotes a vector whose components are the differential coefficients of the components of $\boldsymbol{r}$.

The vector $\boldsymbol{q}$ has a very graphic geometrical significance. For if we take into account (39) the equations (41) and (42) become:

$$q^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \quad . \quad . \quad . \quad (44)$$

and:

$$\cos\lambda = \frac{dx}{ds}, \quad \cos\mu = \frac{dy}{ds}, \quad \cos\nu = \frac{dz}{ds} \quad . \quad . \quad (45)$$

where ds denotes the element of arc of the space curve, taken positive in the direction of the motion. Thus the direction of $\boldsymbol{q}$ coincides with the direction of the element of arc or of the tangent of the orbital curve, and the quantity $q = |\boldsymbol{q}|$ is the speed of the motion along this curve. By (41) and (42) the vector $\boldsymbol{q}$ is, in magnitude and direction, the diagonal of a rectangular parallelepiped the lengths of whose sides are u, v, w.

§ 19. Further, as in § 6, we define the three quantities:

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \ddot{x} = \frac{du}{dt} = \dot{u} \\ \frac{d^2y}{dt^2} &= \ddot{y} = \frac{dv}{dt} = \dot{v} \\ \frac{d^2z}{dt^2} &= \ddot{z} = \frac{dw}{dt} = \dot{w} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (46)$$

and call them the "acceleration components in the direction of the co-ordinate axes" of the point P at the time t. These are the accelerations with which the projections of P on the co-ordinate axes move rectilinearly. Generalizing as before, we differentiate (40) to define the acceleration component of P in any arbitrary direction (ξ', η', ζ'), which is:

$$\frac{d^2x'}{dt^2} = \ddot{x}' = \frac{du'}{dt} = \dot{u}' = \dot{u}\cos\xi' + \dot{v}\cos\eta' + \dot{w}\cos\zeta' \qquad (47)$$

This is the acceleration with which the projection P' of the point P moves rectilinearly in this direction (ξ', η', ζ').

On the basis of this definition we can prove that acceleration is a vector. For if we set:

$$\dot{u}^2 + \dot{v}^2 + \dot{w}^2 = p^2 \quad . \quad . \quad . \quad (48)$$

$$\frac{\dot{u}}{p} = \cos\alpha, \quad \frac{\dot{v}}{p} = \cos\beta, \quad \frac{\dot{w}}{p} = \cos\gamma \quad . \quad . \quad (49)$$

with the limitation that p must be positive and that the direction-angles α, β, γ must lie between 0 and π, then by (47):

$$\dot{u}' = p(\cos\alpha\cos\xi' + \cos\beta\cos\eta' + \cos\gamma\cos\zeta')$$

and by (37):

$$\dot{u}' = p\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where θ denotes the angles between the directions (ξ', η', ζ') and (α, β, γ). Thus the component $\dot{u}'$ is the projection of the distance p, which lies in the direction (α, β, γ), on the x' direction. Analogously to (43) we call this directed quantity the "acceleration vector":

$$\mathbf{p} = \dot{\mathbf{q}} = \ddot{\mathbf{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

From (48) and (49) we see that it is represented in magnitude and direction by the diagonal of a rectangular parallelepiped the lengths of whose sides are $\dot{u}$, $\dot{v}$, $\dot{w}$.

§ 20. The term "acceleration" often misleads beginners to confuse the quantity $|\dot{\mathbf{q}}| = p$ with the quantity $\dot{q} = \frac{dq}{dt}$. It is the same error that would be made if we set

$q(=|\dot{\boldsymbol{r}}|)$ equal to $\frac{dr}{dt}$. Let us therefore investigate the relationship a little more closely. By (41) we get by differentiating with respect to the time:

$$q\dot{q} = u\dot{u} + v\dot{v} + w\dot{w},$$

and hence by (49) and (42):

$$\dot{q} = p(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu)$$

and by (37):

$$\dot{q} = p\cos(\boldsymbol{p}, \boldsymbol{q}) \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Comparing this with (50) we get $\dot{q}$ as the component of the acceleration vector in the direction of the velocity. Since the directions $\dot{\boldsymbol{q}}$ (α, β, γ) and $\boldsymbol{q}$ (λ, μ, ν) are entirely independent of each other, $\dot{q}$ can have any value between $+p$ and $-p$. It is only when these two directions coincide, as in rectilinear motion, that $\dot{q} = p$. But if, for example, the acceleration is perpendicular to the velocity, then $\dot{q} = 0$—that is, the value of the velocity q is constant, whereas the acceleration p may assume any arbitrary (positive) value.

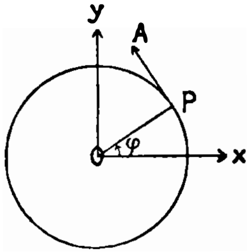

Fig. 4.

§ 21. To illustrate the above definitions and theorems still further we shall consider a special simple case, namely, *uniform motion of a point P in a circle.* Such a motion is represented by the equation:

$$x = r\cos\omega t, \quad y = r\sin\omega t, \quad z = 0 \quad . \quad . \quad (53)$$

where r denotes the radius of the circle, $\omega(>0)$ the angular velocity or radian frequency (§ 12). The orbit of the circle (Fig. 4) is obtained by eliminating t from (53):

$$x^2 + y^2 = r^2, \quad z = 0.$$

The components of the velocity are obtained from (39), if we use $\omega t = \phi$ as an abbreviation, in the form:

$$u = -\omega r\sin\phi, \quad v = \omega r\cos\phi, \quad w = 0.$$

The magnitude and direction of the velocity PA are obtained from (41) and (42) as:

$$q = \omega r,$$

$$\lambda = \frac{\pi}{2} + \phi, \quad \mu = \phi, \quad \nu = \frac{\pi}{2}.$$

The components of the acceleration are, by (46):

$$\dot{u} = -\omega^2 r \cos\phi, \quad \dot{v} = -\omega^2 r \sin\phi, \quad \dot{w} = 0.$$

Finally, the magnitude and direction of the acceleration are, by (48) and (49):

$$p = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\alpha = \pi - \phi, \quad \beta = \frac{\pi}{2} + \phi, \quad \gamma = \frac{\pi}{2}.$$

Thus the acceleration vector is directed from P to the centre O, and here we have an example of the case mentioned at the end of the preceding section in which the direction of the acceleration is perpendicular to the direction of the velocity, which causes q to be constant.

§ 22. At this stage we go a step further and now inquire into the *cause* of a motion. For this purpose we introduce the force that produces the motion. It is obvious that the definition of a force in the case of motion in space must contain that used for rectilinear motion as a special case. Hence, following on (8) we are compelled to set:

$$\left.\begin{aligned} X &= m\dot{u} = m\frac{d^2x}{dt^2} \\ Y &= m\dot{v} = m\frac{d^2y}{dt^2} \\ Z &= m\dot{w} = m\frac{d^2z}{dt^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

where m again denotes the inertial mass of the material point, which is independent of the nature of the motion. Further, following on (47):

$$X' = m\dot{u}' = X\cos\xi' + Y\cos\eta' + Z\cos\zeta' \quad . \quad (56)$$

Hence we define generally the force component in any direction (ξ', η', ζ') as the product of the mass and the component of the acceleration in this direction. From this it follows that force is a vector, whose direction coincides with the direction of the acceleration and whose value differs from that of the acceleration only by a constant factor m. In fact, if we denote the force vector by **F**—that is, set:

$$\boldsymbol{F} = m\dot{\boldsymbol{q}} = m\ddot{\boldsymbol{r}} \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

then we obtain for the absolute value F of this vector from (55) and (48):

$$F^2 = X^2 + Y^2 + Z^2 = m^2p^2 \quad . \quad . \quad . \qquad (58)$$

and for the direction of the vector, from (49) and (55):

$$\cos\alpha : \cos\beta : \cos\gamma = X : Y : Z \quad . \quad . \quad . \qquad (59)$$

and for the component in any arbitrary direction (ξ', η', ζ'), which forms the angle θ with the direction of the force, from (56) and (50):

$$X' = F . \cos\,\theta \quad . \quad . \quad . \quad . \quad . \qquad (60)$$

In virtue of (58) and (59) the force-vector **F** will be represented in magnitude and direction by the diagonal of a rectilinear parellelepiped, the lengths of whose sides are X, Y, Z. Since, as comparison with (56) and (60) shows, these three components may completely replace the force vector **F** and *vice versa*, they are also completely equivalent to it causally—that is, we can compound together three forces X, Y, Z that act in the directions of the co-ordinate axes to form a single force whose value F is determined by (58) and whose direction (α, β, γ) is determined by (59). In the same way we can resolve any arbitrarily directed force according to the same law into three forces that act in the directions of the co-ordinate axes.

§ 23. The equations (55) or (57) contain the fundamental law of the mechanics of a material point. We may either use them, when the motion (38) is known, to determine the force which causes this motion, or, conversely, when the force is known to determine the motion which is caused by the force. The former is, as we see, a problem involving differential calculus, the latter one which involves integral calculus and is hence in general more complicated mathematically.

Let us first discuss a problem of the former kind, by inquiring into the force which causes the uniform circular motion considered in § 21. This comes out directly by combining equations (54) and (58) as :

$$F = m\omega^2 r \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

and is directed, like the acceleration, from P towards the centre O of the circle. Its existence can be demonstrated by swinging the material point P round in a circle at the end of a thread. F then gives us the tension of the thread. The equation (61) may also be written in the form :

$$F = \frac{mq^2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

These two forms of expression (61) and (62) give us a good example of the theorem that the question : "Is F directly or inversely proportional to r in a uniform circular motion ?" has no sense so long as it is not specified whether ω or q is to be regarded as constant in the motion. The same holds for any quantity which depends on more than one variable.

In applying the theory to processes in nature we are mostly concerned with solving the second problem mentioned—namely, to determine the motion when the force is given. Then we have to integrate three differential equations of the second degree, so that six constants of integration occur. These are determined by the "initial state" (§ 10) of the material point—that is, by the position and velocity of the point at the time $t = 0$:

$$\mathbf{r} = \mathbf{r}_0, \quad \mathbf{q} = \mathbf{q}_0 \quad . \quad . \quad . \quad . \quad . \qquad (63)$$

which give us precisely the six required conditions. In general we take the " state " of a material point at any time t to be the complex concept (*Inbegriff*) formed of the six quantities which denote the vector of the position and the vector of the velocity for this time.

Let us take as our first example the simplest case where the force $\boldsymbol{F} = 0$. It then follows from (55) by integrating once and by using the initial state (63) that :

$$\frac{dx}{dt} = u = u_0, \quad \frac{dy}{dt} = v = v_0, \quad \frac{dz}{dt} = w = w_0,$$

and by integrating again :

$$x = u_0t + x_0, \quad y = v_0t + y_0, \quad z = w_0t + z_0 \quad . \qquad (64)$$

The orbital curve is :

$$\frac{x - x_0}{u_0} = \frac{y - y_0}{v_0} = \frac{z - z_0}{w_0},$$

that is, a straight line, which is given by the initial position and the direction of the initial velocity and which is traversed with uniform velocity—in accordance with the law of inertia. From this it follows that whenever a material point moves in a path which is not a straight line, even if the motion is uniform, the presence of a force is indicated. We have already found this confirmed above for the case of uniform circular motion.

§ 24. If several forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, . . . act simultaneously on a material point, they may be replaced by a single force $\boldsymbol{F}$; for in every motion of the point its acceleration has a definite value. This " resultant " force $\boldsymbol{F}$ is found by resolving each of the individual forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, . . . by (58) and (59) into its components :

$$X_1 = F_1 \cos \alpha_1, \quad Y_1 = F_1 \cos \beta_1, \quad Z_1 = F_1 \cos \gamma_1 \quad . \qquad (65)$$

and then combining them by (19) by adding algebraically the components which correspond to a definite co-ordinate direction. In this way the three components :

$$X = \Sigma X_1, \quad Y = \Sigma Y_1, \quad Z = \Sigma Z_1 \quad . \quad . \qquad (66)$$

of the resultant force $\boldsymbol{F}$ result. In vector calculus this composition is briefly denoted by :

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \ . \ . \ . = \Sigma \boldsymbol{F}_1 \quad . \quad . \quad (67)$$

where the " sum of the vectors " or the " vectorial sum " denotes a vector whose components are the algebraic sums of the components of the individual vectors. The absolute value of this sum F which represents the magnitude of the resultant force is, of course, to be carefully distinguished from the sum of the quantities F_1, F_2, F_3, . . . of the individual forces.

§ 25. Before we pass on to consider further applications we shall draw a somewhat clearer picture of the general causal relationship between force and motion. Suppose a given force $\boldsymbol{F}$ of arbitrary magnitude and direction acts on a point which is moving with an arbitrary motion and whose velocity-state is given by the vector $\boldsymbol{q}$. What is the influence of the force on the subsequent motion? If $\boldsymbol{F}$ were equal to zero the point would continue to move in a straight line, but *only* in this case. Hence it follows that $\boldsymbol{F}$ has some definite relationship with the deviation of the motion from uniform motion in a straight line.

Which part of $\boldsymbol{F}$ effects the deviation from uniform motion, that is, which alters the absolute value $|\boldsymbol{q}| = q$, and which part effects the deviation from linear motion—that is, the change of direction (λ, μ, ν) of $\boldsymbol{q}$?

The answer to this question is obtained most simply by the following calculation. If in the equations (55) we replace the quantities u, v, w according to (42) and (45) by $q\frac{dx}{ds}$, $q\frac{dy}{ds}$, $q\frac{dz}{ds}$, then, if we perform the differentiation with respect to t :

$$\left.\begin{aligned} X &= m\frac{dq}{dt}\frac{dx}{ds} + mq^2\frac{d^2x}{ds^2} \\ Y &= m\frac{dq}{dt}\frac{dy}{ds} + mq^2\frac{d^2y}{ds^2} \\ Z &= m\frac{dq}{dt}\frac{dz}{ds} + mq^2\frac{d^2z}{ds^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (68)$$

If we denote the first summands in these equations by X_1, Y_1, Z_1 and the second by X_2, Y_2, Z_2, then by (67) we may regard $\boldsymbol{F}$ as the resultant of two single forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, whose components are represented by these six summands as given above. Hence:

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

The first single force $\boldsymbol{F}_1$ has the absolute value:

$$F_1 = \sqrt{X_1^2 + Y_1^2 + Z_1^2} = m\left|\frac{dq}{dt}\right| \quad . \qquad (70)$$

and its direction coincides with the direction of the element of arc ds or the velocity $\boldsymbol{q}$.

The second single force $\boldsymbol{F}_2$ has the absolute value:

$$F_2 = mq^2\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \quad . \qquad (71)$$

and its direction ratios are:

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} \quad . \quad . \quad . \quad . \quad . \qquad (72)$$

The two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are mutually perpendicular, for:

$$\frac{dx}{ds}\cdot\frac{d^2x}{ds^2} + \frac{dy}{ds}\cdot\frac{d^2y}{ds^2} + \frac{dz}{ds}\cdot\frac{d^2z}{ds^2} = 0 \quad . \quad . \qquad (73)$$

as we find by differentiating the identity:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1 \quad . \quad . \qquad (73a)$$

with respect to s. Hence we have here resolved the force $\boldsymbol{F}$ into two components $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, the first of which acts in the direction of the motion and the second in a direction perpendicular to the direction of motion; and of all the normals to the curve the direction (72) is that of the "principal normal" or the normal which lies in the "osculating plane" of the curve (*Krümmungsebene*)—that is, in the plane which has three successive infinitely close points in common with the curve. These three points also determine the circle which approaches the curve most closely and whose radius is therefore called the "radius of curvature" ρ of the curve. Its reciprocal value is equal to the square root in (71):

$$\frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \quad . \quad . \quad (74)$$

All these results can be combined in the following physical statement: to find the influence of any arbitrarily given force $\boldsymbol{F}$ on a material point which is moving with an arbitrarily given velocity $\boldsymbol{q}$ we resolve the force $\boldsymbol{F}$ into the two components parallel and perpendicular to the direction of the velocity. The absolute value F_1 of the first component, the "tangential force" $\boldsymbol{F}_1$ gives, by (70), the change in the magnitude of the velocity:

$$\frac{dq}{dt} = \pm \frac{F_1}{m} \quad . \quad . \quad . \quad . \quad (74a)$$

where the + or the − sign applies according as the direction of $\boldsymbol{F}_1$ is in the same direction as the velocity or in the opposite direction. The direction of the second component, the "normal force" $\boldsymbol{F}_2$, gives us the principal normal to the orbital curve and hence the osculating plane; its absolute value F_2 gives, by (71) and (74), the radius of curvature:

$$\rho = \frac{mq^2}{F_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Since the normal force is directed towards the centre of the circle of curvature it is often called the "centripetal force." This term is not entirely free from objection, as it easily gives the impression that this force acts in the direction of a prescribed objective, the centre of curvature. But the actual state of affairs is just the reverse: what is primarily given is the force $\boldsymbol{F}_2$, and the curvature is only secondary, being produced by the force; by (75) the curvature depends not only on the force, but also on the velocity-state of the moving point. The more rapidly the point is moving the greater is ρ and the smaller is the curvature.

The reader who is not familiar with the analytical relations that have here been used and that involve the principal normal and the radius of curvature of a space-curve may derive the above mechanical theorems in the following more geometrical way. From (56) we get by differentiating with respect to t and taking into account (42a):

$$X' = m\frac{d}{dt}(q\cos\epsilon) = m\cdot\frac{dq}{dt}\cos\epsilon - mq\sin\epsilon\frac{d\epsilon}{dt} \quad . \qquad (76)$$

where X' denotes the component of the force $\boldsymbol{F}$ in any arbitrarily chosen fixed direction x' and ϵ is the angle which this direction forms with the direction of the velocity $\boldsymbol{q}$.

According as we make the constant direction x' coincide with the tangent $\epsilon = 0$ or the principal normal $\left(\epsilon = \frac{\pi}{2}, d\epsilon = -\frac{ds}{\rho}\right)$ or the bi-normal $\left(\epsilon = \frac{\pi}{2}, d\epsilon = 0\right)$ at a definite point of the space-curve we get from (76) either the tangential force or the normal force or zero, and hence the preceding theorems follow.

The relationship between the tangential force and the acceleration component $\frac{dq}{dt}$ is clearly a generalization of the law (8) which governs rectilinear motion, and that between the normal force and the radius of curvature ρ is a generalization of the law (62) which governs uniform circular motion.

§ 26. We next consider the motion of a material point under the influence of its gravitational force alone; this is the same problem as that treated in § 10 except that now the initial velocity q_0 of the point need not be in a vertical direction, but may form any arbitrary angle $\lambda_0 \left(<\frac{\pi}{2}\right)$ with the horizontal. The motion clearly follows in that vertical plane which is defined by the direction of the initial velocity. If we now choose (as we shall make a rule of doing in the sequel) the z-axis to be in the upward direction, the x-axis in the plane of the motion and the origin of co-ordinates at the initial position of the point, then the complete equations of motion (55) run :

$$X = 0 = m \cdot \frac{du}{dt}, \quad Z = -mg = m \cdot \frac{dw}{dt} \quad . \quad (76a)$$

with the initial conditions, for $t = 0$:

$$x_0 = 0, \quad z_0 = 0, \quad u_0 = q_0 \cos \lambda_0, \quad w_0 = q_0 \sin \lambda_0$$

Integrating once and taking into account the initial conditions, we get :

$$\left.\begin{aligned} u &= \frac{dx}{dt} = q_0 \cos \lambda_0 \\ w &= \frac{dz}{dt} = -gt + q_0 \sin \lambda_0 \end{aligned}\right\} \quad . \quad . \quad (77)$$

Integrating again we have :

$$\left.\begin{aligned} x &= q_0 \cos \lambda_0 \, . \, t \\ z &= -\frac{1}{2} g t^2 + q_0 \sin \lambda_0 \, . \, t \end{aligned}\right\} \quad . \quad . \quad (78)$$

Eliminating t from these two equations we get the equation of the orbital curve :

$$z = -\frac{g}{2 q_0^2 \cos^2 \lambda_0} \cdot x^2 + \tan \lambda_0 \, . \, x \quad . \quad . \quad (79)$$

a parabola whose axis is parallel to the z-axis (Fig. 5).

Its second point of intersection with the x-axis gives the horizontal range or distance of throw (*Wurfweite*) :

$$OA = \frac{q_0^2 \sin (2\lambda_0)}{g}$$

The height of its vertex above the x-axis (*Wurfhöhe*) is given by $\frac{dz}{dx} = 0$:

$$BC = \frac{q_0^2 \sin^2 \lambda_0}{2g}$$

If we assume q_0 as constant but λ_0 as variable, we get the greatest range when $\lambda_0 = \frac{\pi}{4}$; and the greatest height is attained when $\lambda_0 = \frac{\pi}{2}$. The highest point reached is technically called the culminating point.

To strike a definite point (x_1, z_1) for a given definite q_0 we must choose λ_0 so that the equation (79) is satisfied by $x = x_1$, $z = z_1$. This gives us a quadratic equation for $\tan \lambda_0$, and hence again either two or no real values for λ_0 except in the limiting case. Hence for a given initial velocity either the target-point can be struck by aiming in two different directions [*Flachschuss* (low or direct aim), *Steilschuss* (high or indirect aim)] or, if it is too far away, by no aim at all.

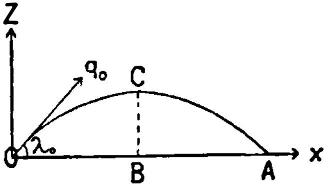

FIG. 5.

Another noteworthy relationship is that which tells us how great the velocity q of the point must be at a definite height z. This is obtained by eliminating t from (77) and (78) and comes out very simply as:

$$q^2 + 2gz = q_0^2 \quad . \quad . \quad . \qquad (80)$$

The velocity q does not therefore depend on x, but only on the height z, and the parabola does not only lie symmetrically with regard to its axis, but is also traversed symmetrically, since at the two points for which z has the same value, the velocity is again the same.

§ 27. If we wish to take into account the resistance of the air we must introduce a second force in addition to gravity, and this force is directed at every moment oppositely to the velocity at that moment; also its magnitude W depends in a certain way on q. The equations of motion are then, in view of (66):

$$\left.\begin{aligned} X &= -W\frac{dx}{ds} = -W\frac{u}{q} = m\frac{du}{dt} \\ Z &= -mg - W\frac{dz}{ds} = -mg - W\frac{w}{q} = m\frac{dw}{dt} \end{aligned}\right\} \quad . \qquad (81)$$

These equations can be integrated only if W is given as a function of q. For smaller velocities W is proportional to q (cf. § 13), for greater velocities it is found experimentally that W varies more rapidly with q than its first power. The orbit no longer comes out as a parabola but as a "ballistic curve." If this curve is evaluated in a particular case it can serve conversely to determine W as a function of q.

CHAPTER III

CENTRAL FORCES. POTENTIAL

§ 28. BEFORE passing on to integrate the equations of motion of a material point we must first find out what force is acting on it. The present chapter is devoted to this problem. Among all forces in nature those which have been most investigated are central forces (§ 12), and among them the most important again are those whose magnitude is inversely proportional to the square of the distance, as in Newtonian gravitation. We shall therefore deal with the inverse square law first. We may leave out of the discussion entirely the question as to the origin of gravitation; for the significance of the law of gravitation does not depend on the answer to this question but on the circumstance that it comprehends the motions of all the heavenly bodies to the smallest details in one very simple and very accurate expression.

According to Newton's law of gravitation a material point of mass m is attracted by another material point of mass μ and at a distance r with the force:

$$F = f \cdot \frac{m\mu}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Here f, the gravitational constant, denotes an absolute or universal constant whose numerical value of course depends on the units fixed for length, mass and time; moreover, by (8a), f has the dimensions:

$$\left[\frac{l^3}{mt^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

We shall calculate the numerical value of f later in cms., grms. and secs. (§ 34).

If we had not already arbitrarily fixed the unit of mass there would have been nothing to prevent our choosing the unit of mass so that:

$$f = 1.$$

The gravitational constant would then be a pure number and the mass would not be a self-dependent quantity but would have the dimensions:

$$\left[\frac{l^3}{t^2}\right].$$

The unit of mass so defined is used for convenience in astronomy and is called the "astronomic unit of mass." From this we again see that the dimensions of a physical quantity are not inherent in it, but constitute a conventional property conditioned by the choice of the system of measurement. If this circumstance had always been properly appreciated, a great number of unfruitful controversies in physical literature, particularly concerning that of the electromagnetic system of measurement, would have been avoided.

§ 29. The expression (82) not only gives us the force with which the point m is attracted by the point μ, but it also represents the force with which the point μ is attracted by the point m, as can be inferred at once from the symmetrical form of the expression. This is a special case of Newton's Third Law, the Principle of Action and Reaction, which states in its general form: *to every force which one material point exerts on a second point there is an equally great and oppositely directed force which is exerted on the first point by the second.* A stone of weight G which falls to earth attracts the earth with the same force G as the earth exerts on the stone. The fact that the earth does not move appreciably towards the stone is only due to the inertial mass of the earth being enormously greater than that of the stone, so that by (8) the acceleration of the earth due to the force G would be vanishingly small.

Newton's third law can be traced back quite generally to other principles (cf. § 129). The following remarks will suffice for the special case under consideration. We imagine the two material forces m and μ to be "rigidly connected"—that is, to be fastened to the ends of an incompressible and inextensible rod of vanishingly small mass, but free to move about. Now if the attractive forces exerted on the two points by each other, indicated by arrows in Fig 6, were not equal, the whole system under consideration would have to start moving in the direction of the greater force, and since the distance r and hence also the forces remain constant, the difference would also remain constant and hence the velocity of the rod would in the course of time increase beyond all limits. Such a process is impossible in nature.

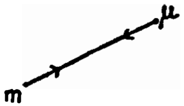

FIG. 6.

§ 30. If we denote the co-ordinates of the point m (Fig. 6) by x, y, z and those of the point μ by ξ, η, ζ, then the components of the gravitational force which acts on m are:

$$\left.\begin{aligned} X &= f\frac{m\mu}{r^2}\cdot\frac{\xi - x}{r} \\ Y &= f\frac{m\mu}{r^2}\cdot\frac{\eta - y}{r} \\ Z &= f\frac{m\mu}{r^2}\cdot\frac{\zeta - z}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad (84)$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 \quad . \quad . \quad (85)$$

As a consideration of simple special cases easily convinces us, these expressions also give us the correct sign for the components for all positions of the two points, if the magnitude r is always taken as positive.

If the point m is attracted simultaneously by several points whose masses are $\mu_1, \mu_2, \mu_3 \ldots$, the components of the resultant force which acts on it are, by (66) and (84):

$$X = fm\,.\,\Sigma\frac{\mu_1(\xi_1 - x)}{r_1{}^3} \text{ and so forth} \quad . \quad . \quad (86)$$

where the summation is to be taken over the indices 1, 2, 3, . . .

§ 31. We shall now assume that the attracting masses occupy a finite space continuously—that is, we shall calculate the gravitational action of a continuously extended material body on a material point. This problem may be reduced to the preceding problem by dividing the material body by means of an (infinity)^3 family of planes parallel to the co-ordinate planes into an (infinity)^3 number of volume elements, each of which contains a mass μ which may be regarded as a material point. To find μ we first assume the body to be "homogeneous"—that is, that it contains equal masses in equal volumes. Then the ratio of any part of the mass to the volume which it occupies is a constant, and is equal to the quotient of the mass M of the whole body by its volume V:

$$\frac{M}{V} = k$$

The constant k is the "density" of the homogeneous body. But if the body is *not* homogeneous the ratio of any part ΔM of the mass to the volume ΔV which it occupies is called the mean density of the body in the volume in question. The mean density depends in general on the position, size and shape of the volume considered. If we now allow the volume ΔV to decrease without limit until it becomes the volume dV, in which process the contained mass also shrinks to a material point μ, the mean density merges into the local density:

$$\frac{\mu}{dV} = k \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (87)$$

which now depends only on the position ξ, η, ζ and not on the size and form of the element of volume:

$$dV = d\xi \, . \, d\eta \, . \, d\zeta \qquad . \quad . \quad . \quad . \qquad (88)$$

If k is given as a function of ξ, η, ζ the distribution of mass in the whole body is completely determined. In particular, the total mass of the body is, by (87):

$$M = \Sigma\mu = \int k dV \quad . \quad . \quad . \quad . \quad (89)$$

By substituting the value of μ from (87) in (86) we get the components of the attractive force which the material point m experiences owing to the action of a continuously extended material body of given density k:

$$X = fm \int \frac{(\xi - x)k dV}{r^3} \text{ and so forth} \quad . \quad . \quad (90)$$

where:

$$r = + \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \quad . \quad (91)$$

The integration is to be performed over all the points ξ, η, ζ of the body, where k is to be regarded as a given function of ξ, η, ζ, whereas the quantities x, y, z remain constant during the integration.

§ 32. As an illustration we shall calculate the attraction which a material sphere of given density k exerts on the material point m. To perform the integration in (90) it is expedient to introduce in place of the rectilinear co-ordinates ξ, η, ζ the polar co-ordinates ρ, θ, ϕ, whose meaning can be exemplified as follows from Fig. 3 (§ 17). If the polar co-ordinates ρ, θ, ϕ refer to the point P then ρ (positive) is the distance OP, θ (between 0 and π) is the angle between the z-axis and the direction OP, and ϕ (between 0 and 2π) is the angle between the xz-plane AOz and the plane BOz which contains the point P, measured in the direction from the xz-plane to the yz-plane. From this we arrive uniquely at the relationships between the polar co-ordinates and the rectilinear co-ordinates ξ, η, ζ of the point P:

$$\xi = \rho \sin\theta \cos\phi, \quad \eta = \rho \sin\theta \sin\phi, \quad \zeta = \rho \cos\theta \quad . \quad (92)$$

We also divide the body into elements of volume dV which correspond to the polar co-ordinates that have been introduced. First we divide the whole sphere into infinitely thin concentric spherical layers, one of which has the internal radius ρ and the external radius $\rho + d\rho$, and we first calculate the attraction of the mass contained in this spherical layer on the point m—that is, we take the integration in (90) only over the volume-element dV of this spherical layer. Then ρ and $\rho + d\rho$ remain constant during this integration and we have only to integrate over θ and ϕ. To express dV in polar co-ordinates we subdivide the spherical layer further by means of an infinite number of infinitely near surfaces $\theta =$ const. and $\phi =$ const. The former are simple circular cones described with their centre at O around the z-axis; the latter are half-planes which are bounded by the z-axis. Two adjacent cones θ and $\theta + d\theta$ cut two parallel meridians of latitude out of the sphere ρ, two neighbouring planes ϕ and $\phi + d\phi$ cut two half meridians of longitude out of the sphere ρ (Fig. 7). These four lines mark off a rectangular element of surface, whose area is represented by the product of the element of arc on the meridian of longitude, $\rho . d\theta$, and the element of arc on the meridian of latitude, $\rho \sin \theta . d\phi$. Multiplying this element of area by $d\rho$ we get as the volume-element of the spherical layer:

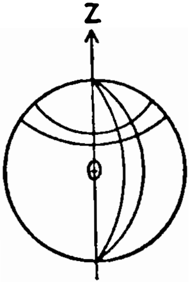

FIG. 7.

$$dV = \rho^2 \sin\theta \,.\, d\theta \,.\, d\phi \,.\, d\rho \quad . \quad . \quad . \quad (93)$$

To simplify the calculation we assume the attracted point m to lie on the positive z-axis, which in no wise restricts the generality of the argument—that is:

$$x = 0, \quad y = 0, \quad z > 0 \quad . \quad . \quad . \quad . \quad (94)$$

If we assume the spherical layer to be homogeneous—that is, assume the density k to be independent of θ and ϕ, then, as we can easily see on physical grounds, $X = 0$, $Y = 0$, and the whole attraction of the spherical layer or shell reduces to the component Z, which by (90) and in view of (92) and (93) comes out as:

$$Z = fmk\rho^2 d\rho \cdot \iint \frac{\rho \cos\theta - z}{r^3} \cdot \sin\theta \cdot d\theta \, d\phi.$$

The integration over ϕ from 0 to 2π may be performed and gives :

$$Z = 2\pi f m k \rho^2 \, d\rho \cdot \int_0^\pi \frac{\rho \cos\theta - z}{r^3} \cdot \sin\theta \, d\theta.$$

The last integral may be easily evaluated if we use r as the variable of integration. Here we have, by (91), (92) and (94) :

$$r^2 = \rho^2 + z^2 - 2\rho z \cos\theta \quad . \quad . \quad . \quad (95)$$

and hence, when ρ and z are constant :

$$r dr = \rho z \sin\theta d\theta \quad . \quad . \quad . \quad . \quad (96)$$

Consequently, if we introduce r and dr in place of θ and $d\theta$:

$$Z = 2\pi f m k \rho d\rho \cdot \int_{r_0}^{r_1} \left(\frac{\rho^2 + z^2 - r^2}{2z} - z \right) \cdot \frac{dr}{zr^2}.$$

Here r_0 and r_1 are the values of r for $\theta = 0$ and $\theta = \pi$, and so by (95), since $r > 0$:

$$r_0 = |z - \rho|, \quad r_1 = z + \rho \quad . \quad . \quad (97)$$

If we perform the integration we get :

$$Z = -\frac{\pi f m k \rho d\rho}{z^2} \cdot \left\{ r_1 - r_0 - (\rho^2 - z^2)\left(\frac{1}{r_0} - \frac{1}{r_1}\right)\right\}.$$

To be able to evaluate r_0 we must distinguish between two cases.

Case I. $z > \rho$; that is, the point m is situated outside the spherical layer.

Then $r_0 = z - \rho$, and :

$$Z = -\frac{4\pi f m k \rho^2 d\rho}{z^2} = -\frac{fm \, . \, dM}{z^2} \quad . \quad . \quad (97a)$$

where we use dM to denote the total mass of the spherical layer. Hence the attraction exerted by a homogeneous spherical layer on a material point outside it is just the same as if the mass of the layer were concentrated at the centre of the sphere.

Case II. $z < \rho$; that is, the point m is situated inside the hollow space of the spherical layer.

Then $r_0 = \rho - z$, and :

$$Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97b)$$

The attraction exerted by a homogeneous spherical layer or shell on a material point inside it is thus equal to zero.

§ 33. The results which have been obtained for the gravitational action of an infinitely thin homogeneous spherical layer may be applied directly to calculate the attraction of a spherical layer of finite thickness; the solid sphere is included as a particular case. It is only necessary to assume that the density k is independent of the angles θ and ϕ, whereas it may depend arbitrarily on the radius vector ρ.

We shall restrict ourselves here to considering the attraction exerted by a hollow homogeneous sphere of radii ρ_1 and $\rho_2(>\rho_1)$, on a point-mass m situated at a distance r_0 from the centre of the sphere. Three cases are to be distinguished.

Case I. The point m lies outside the hollow sphere. $r_0>\rho_2$.

Then each of the concentric and infinitely thin spherical layers acts as if its mass were concentrated at the centre of the sphere. Hence the whole mass of the sphere:

$$M = \frac{4}{3}\pi(\rho_2{}^3 - \rho_1{}^3)k$$

acts in the same way and the attraction becomes:

$$F_1 = \frac{4}{3}\pi f m k \cdot \frac{\rho_2{}^3 - \rho_1{}^3}{r_0{}^2} \quad . \quad . \quad . \quad (98)$$

Case II. The point m lies somewhere inside the hollow sphere. $r_0<\rho_1$.

The attraction is then:

$$F_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Case III. The point m lies inside the mass of the layer. $\rho_2 > r_0 > \rho_1$.

In this case we describe a concentric sphere which passes through the point m, its radius being r_0. This sphere divides the hollow sphere into two parts, an internal hollow sphere of radii ρ_1 and r_0, and an external hollow sphere of radii r_0 and ρ_2. The attraction due to the latter is zero, by (99), so that only the attraction of the former remains, which is expressed, according to (98), by :

$$F_3 = \frac{4}{3}\pi f m k \cdot \frac{{r_0}^3 - {\rho_1}^3}{{r_0}^2} \quad . \quad . \quad . \quad (100)$$

It is interesting to investigate how the value F of the attraction varies when the attracted point m is brought from an infinite distance ($r_0 = \infty$) up to and into the interior of the sphere until $r_0 = 0$. First the formula (98) holds, then the formula (100) and finally the formula (99). It is of particular importance that for the limiting cases $r_0 = \rho_2$ and $r_0 = \rho_1$ the successive formulæ should each time give the same value for F, namely for $r_0 = \rho_2$ the value :

$$F_1 = F_3 = \frac{4}{3}\pi f m k \cdot \frac{{\rho_2}^3 - {\rho_1}^3}{{\rho_2}^2} \quad . \quad . \quad (101)$$

and for $r_0 = \rho_1$ the value :

$$F_3 = F_2 = 0.$$

Thus the attractive force varies *continuously* throughout with the position of the point m, even when the point passes through the surface of the attracting masses. This theorem is clearly valid generally, even for non-spherical masses; for if the attraction F at the surface of an attracting mass were discontinuous—that is, if it had different values on the two sides of the surface—the abrupt transition could be due only to the gravitational action of those particles of mass which lie on the surface and are nearest to m; but these particles can be imagined to a sufficient degree of approximation to be a portion of a homogeneous sphere of appropriate density, and for this case continuity was proved above.

For a solid sphere of radius R we have $\rho_1 = 0$, $\rho_2 = R$, and the attraction exerted on a point m at the distance $r_0 < R$ from the centre is, by (100) :

$$F = \frac{4}{3}\pi f m k r_0 \quad . \quad . \quad . \quad . \quad (102)$$

That is, the attraction of a homogeneous sphere on a point-mass in its interior is directly proportional to its distance from the centre and is independent of the radius of the sphere. The attraction attains its greatest value at the surface of the sphere.

§ 34. An important example of the laws above obtained is given by the gravitational action of the earth on a point-mass m situated at its surface; for this represents the weight $G = mg$ of the point-mass (§ 10). Although the earth is certainly not homogeneous its density k will depend essentially only on ρ and not on the direction (θ, ϕ), so that we may imagine the earth's sphere to be divided into concentric and infinitely thin homogeneous layers. Accordingly its attractive force on a point m at its surface is:

$$\frac{fmM}{R^2} = mg,$$

where R denotes the radius and M the mass of the earth, or:

$$\frac{fM}{R^2} = g \quad . \quad . \quad . \quad . \qquad (103)$$

Here g and R are to be considered directly measurable, so that we can obtain the value of fM. On the other hand, the two factors fM cannot be separated without a special measurement being made. The problem of determining the mass M of the earth is therefore essentially identical with the problem of determining the gravitational constant f. This is solved by measuring the gravitational action of any known mass, for example of a mountain of known form and density, or of a block of lead. The mass of the earth is usually given by specifying its mean density:

$$k_m = \frac{M}{V} = 5{\cdot}5 \left[\frac{\text{grm.}}{\text{cm.}^3}\right] \quad . \quad . \quad . \qquad (104)$$

Since the density of the rock masses which lie at the surface of the earth amounts to about 2·5, the density of the earth increases rapidly towards the centre of the earth.

The numerical value (104) corresponds, by (103), to the value of the gravitational constant:

$$f = 6{\cdot}7 \,.\, 10^{-8}\left[\frac{\text{cm.}^3}{\text{grm. sec.}^2}\right] . \quad . \quad . \quad (105)$$

§ 35. We now revert to the general case of any arbitrary central forces and consider the resultant attraction exerted by a system of masses in arbitrary positions on an individual point-mass P. Since in this portion of the present volume we are concerned with the motion of a single material point, we assume the attracting point-masses, whose co-ordinates we again take as ξ, η, ζ, to be at rest, but the point P, on which the attraction acts and which is therefore called the "reference-point" (*Aufpunkt*) as capable of motion; that is, its co-ordinates x, y, z may vary in value. The question now is: how does the attraction depend, in magnitude and direction, on the position of the reference-point P, that is, on its co-ordinates?

For the sake of generality we shall not take the special Newtonian law of gravitation but any arbitrary law of attraction by setting the value of the attraction which a point ξ, η, ζ exerts on the point P equal to any function $f(r)$ of the distance. For the Newtonian law of attraction $f(r)$ then becomes (82). For the more general law of force the components of the resultant attraction exerted by a system of point-masses at ξ, η, ζ on the point P are, on the model of the equations (84) and (86):

$$\left.\begin{aligned} X &= \Sigma f(r_1) \cdot \frac{\xi_1 - x}{r_1} \\ Y &= \Sigma f(r_1) \cdot \frac{\eta_1 - y}{r_1} \\ Z &= \Sigma f(r_1) \cdot \frac{\zeta_1 - z}{r_1} \end{aligned}\right\} \quad . \quad . \quad . \quad (106)$$

The summation is to be performed over the order numbers 1, 2, 3 of the attracting masses. The case of repulsive forces is also contained in these equations; for then we only need to take f as negative. If the acting masses are distributed continuously over a finite space, then integrals occur in the place of the sums, as above in § 31. The following considerations also apply in that case.

§ 36. To find the influence of a change of position of the reference-point P on the magnitude and direction of the attractive force which acts on it we must investigate the components X, Y, Z of the resultant force as functions of the co-ordinates x, y, z of the reference-point. We then arrive at the important result that the three functions x, y, z can always be referred back to a single function.

For if we set:

$$\int f(r) \, . \, dr = F(r) \quad . \quad . \quad . \quad . \qquad (107)$$

and:

$$U = F(r_1) + F(r_2) + \ . \ . \ . = \Sigma F(r_1) \quad . \qquad (108)$$

we get the following result by differentiating with respect to x, say, the function U so defined:

$$\frac{\partial U}{\partial x} = \Sigma \frac{\partial F(r_1)}{\partial r_1} \cdot \frac{\partial r_1}{\partial x} = \Sigma f(r_1) \cdot \frac{\partial r_1}{\partial x}.$$

Now, by (85) we have if we differentiate partially with respect to x:

$$r_1 \frac{\partial r_1}{\partial x} = x - \xi_1 \quad . \quad . \quad . \qquad (109)$$

If this is substituted in the last equation, we get, in view of (106), the relationship:

$$X = -\frac{\partial U}{\partial x}, \text{ and similarly, } Y = -\frac{\partial U}{\partial y}, \text{ and } Z = -\frac{\partial U}{\partial z} \quad . \qquad (110)$$

The function U, whose negative derivatives with respect to x, y, z represent the force components, is called the *potential* of the masses acting on the point P. An additive constant remains undetermined in it on account of the indefinite lower limit of the integral (107). This constant clearly has no physical significance.

For the special case of Newtonian gravitation $f(r)$ becomes (82) and accordingly, by (107):

$$F(r) = -f\frac{m\mu}{r},$$

and the gravitational potential is, according to (108):

$$U = -fm\Sigma\frac{\mu_1}{r_1}. \quad . \quad . \quad . \quad . \quad (111)$$

Here, for the sake of simplicity, we have given the additive constant a value such that U vanishes when the reference-point P moves to an infinite distance from all the attracting masses.

Through the introduction of potential the treatment of the whole problem of attraction becomes enormously simplified, since now only one function need be found instead of three. Moreover, the potential has several advantages over force itself; for example, it is simple and symmetrical in its structure, and in compounding the effects of several masses the potentials simply add up algebraically, whereas the forces must first be resolved into their components. Quantities such as the potential U and the mass m which have no direction, but which are defined completely by a single numerical value, are called scalar quantities or " scalars," to distinguish them from vector quantities.

If the attracting masses are continuously distributed in space with a density which is given as a function of ξ, η, ζ, then by (87) the sum (111) becomes transformed into the integral:

$$U = -fm\int\frac{kdV}{r} \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Here dV is given by (88) and r by (91), and the integration is to be performed over all the points ξ, η, ζ of the space occupied by the attracting masses. Since $r > 0$ always, the gravitational potential U is an essentially negative quantity.

§ 37. To bring out clearly the advantage of introducing the potential we shall now discuss the same example as above in § 32: the attraction exerted by an infinitely thin homogeneous spherical layer on a reference-point situated on the positive z-axis; but now we shall use the potential. The notation remains the same as formerly. Then by (112) and (93):

$$U = -fmk\rho^2 d\rho \int\int \frac{\sin\theta d\theta d\phi}{r}.$$

The integration with respect to ϕ, from 0 to 2π, can be performed directly. Instead of θ we again introduce r, by means of (95) and (96) and we then easily obtain:

$$U = -2\pi fmk\rho d\rho \cdot \frac{r_1 - r_0}{z}$$

where r_0 and r_1 are given by (97).

We must now again distinguish between two cases:

Case I. $z > \rho$—that is, the reference-point is situated outside the spherical layer. Then $r_0 = z - \rho$ and:

$$U = -\frac{4\pi fmk\rho^2 d\rho}{z} = -f\frac{m \, . \, dM}{z} \quad . \quad (113)$$

where dM again denotes the mass of the spherical layer. Thus the potential of a homogeneous spherical layer for a point-mass outside it is exactly the same as if the mass of the layer were concentrated at the centre of the sphere; and by (110) the attractive force becomes:

$$Z = -\frac{\partial U}{\partial z} = -f\frac{m \, . \, dM}{z^2}$$

which agrees with (97a).

Case II. $z < \rho$—that is, the reference-point lies inside the hollow space. Then $r_0 = \rho - z$, and:

$$U = -4\pi fmk\rho d\rho = -f\frac{m \, . \, dM}{\rho} \quad . \quad . \quad (114)$$

Thus the potential of a homogeneous spherical layer with respect to a point-mass in the interior of the hollow space is independent of its distance from the centre and is exactly as great as if the point-mass were situated at the centre. In that case it is equally distant from all elements of mass of the layer, and the potential, which is the sum of the individual potentials, is obtained simply by dividing the total mass dM of all the elements by the common distance ρ.

For the attractive force we again obtain by (107):

$$Z = -\frac{\partial U}{\partial z} = 0,$$

which agrees with (97*b*).

§ 38. Let us now also calculate the potential of a hollow homogeneous sphere of radii ρ_1 and $\rho_2 (> \rho_1)$ with respect to a point-mass m at a distance r_0 from the centre. Here again, as in § 33, three cases are to be distinguished.

Case I. The point m lies outside the hollow sphere. $r_0 > \rho_2$. Then each of the concentric and infinitely thin spherical layers acts exactly as if its mass were concentrated at the centre. Hence the whole mass of the hollow sphere acts in the same way, and the potential becomes:

$$U_1 = -\frac{4\pi}{3} fmk \cdot \frac{\rho_2^3 - \rho_1^3}{r_0} . \qquad (115)$$

Case II. The point m lies somewhere in the interior of the hollow space. $r_0 < \rho_1$.

Then the potential is independent of r_0 and exactly as great as if m were situated at the centre—namely, by (114), if we integrate over all the spherical layers:

$$U_2 = -4\pi fmk \int_{\rho_1}^{\rho_2} \rho d\rho,$$

$$U_2 = -2\pi fmk (\rho_2^2 - \rho_1^2) \quad . \quad . \quad . \qquad (116)$$

Case III. The point m lies inside the layer of the mass itself. $\rho_2 > r_0 > \rho_1$.

We describe through the point m the concentric spherical surface of radius r_0, which divides the whole hollow sphere into two parts, an inner hollow sphere of radii ρ_1 and r_0

and an outer hollow sphere of radii r_0 and ρ_2. The potential of the inner hollow sphere results from (115), that of the outer from (116), the respective radii being taken into account. Thus the required potential, being the sum of the two partial potentials, is :

$$U_3 = -\frac{2\pi}{3} fmk\left(3\rho_2{}^2 - \frac{2\rho_1{}^3}{r_0} - r_0{}^2\right) \qquad (117)$$

If we move the reference-point m from an infinite distance ($r_0 = \infty$) up to the hollow sphere and through the layer of matter into the inner hollow sphere, first the formula (115) holds, then (117) and finally (116). For the limiting cases $r_0 = \rho_2$ and $r_0 = \rho_1$ the successive formulæ in each case give the same value for U; namely : for $r_0 = \rho_2$ the value is :

$$U_1 = U_3 = -\frac{4\pi}{3} fmk\left(\rho_2{}^2 - \frac{\rho_1{}^3}{\rho_2}\right) \quad . \quad . \quad (118)$$

and for $r_0 = \rho_1$ the value is :

$$U_3 = U_2 = -2\pi fmk\,(\rho_2{}^2 - \rho_1{}^2).$$

Thus the potential U changes *continuously* with the position of the reference-point m, even when m passes through the surface of the attracting mass, and it is easy to see from considerations similar to those adduced in § 33 that this theorem also holds for non-spherical and non-homogeneous masses.

For a solid sphere of radius R we have $\rho_1 = 0$, $\rho_2 = R$, and the gravitation potential at a point m at a distance $r_0 < R$ from the centre is, by (117) :

$$U = -\frac{2\pi}{3} fmk\,(3R^2 - r_0{}^2) \quad . \quad . \quad . \quad (119)$$

§ 39. Let us now consider more closely the physical significance of potential; we shall not restrict ourselves to gravitation, but, in conformity with the expression (108), we shall assume any arbitrary law of attraction. Corresponding to every point x, y, z of space regarded as a reference-point there is a definite value of the potential

U, and by equation (110) each of the three force-components acts in the direction in which U decreases; if, for example, U increases in the positive direction of x, then X is negative. Moreover, the force-component is the greater, the more rapidly U varies with the co-ordinate in question; it is equal to the "potential gradient" in the direction in question. This may also be expressed by saying the attractive force tends to decrease the potential U.

Since any direction in space may be chosen as a co-ordinate axis, we also have that for any arbitrary direction x':

$$X' = -\frac{dU}{dx'} \quad . \quad . \quad . \quad . \quad . \qquad (120)$$

To show this in a more analytical way we form the differential coefficient:

$$\frac{dU}{dx'} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dx'} + \frac{\partial U}{\partial y}\frac{dy}{dx'} + \frac{\partial U}{\partial z}\frac{dz}{dx'} \quad . \qquad (120a)$$

Here the second factors of the three products are the cosines of the angles ξ', η', ζ', which the direction x' makes with the co-ordinate axes, as may be seen from Fig. 8, where $PA = dx'$, $PB = dx$, and the angle $P = \xi'$. Thus, taking into account (110) we have:

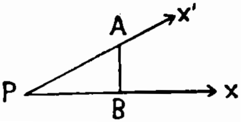

FIG. 8.

$$\frac{dU}{dx'} = -(X\cos\xi' + Y\cos\eta' + Z\cos\zeta'),$$

from which, by (56), we immediately get (120).

The preceding discussion at the same time shows quite generally that the differential coefficient of any scalar function U of x, y, z with respect to the different directions of space always represents the components of a vector, which is called the space "gradient of U" and is written: grad U. We may therefore write equations (110) in vectorial language briefly as:

$$\boldsymbol{F} = -\operatorname{grad} U \quad . \quad . \quad . \quad . \quad . \qquad (121)$$

§ 40. The way in which the attracting force ***F*** depends in magnitude and direction on the position of the reference-point is brought out most clearly by a graphical method of representation. Let us imagine that at every reference-point the corresponding value of the potential has been noted and let us take together all those reference-points which have a definite value, say c, for the potential. The co-ordinates of these points then satisfy the equation :

$$U = c,$$

That is, the points form a surface, which is called an "equipotential surface." Corresponding to every value of the constant c there is a definite equipotential surface and by varying c from $-\infty$ to $+\infty$ we obtain all the possible equipotential surfaces which fill the whole of infinite space. A level surface may also consist of several shells entirely distinct from one another, but two different equipotential surfaces can never intersect.

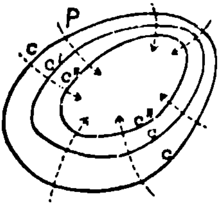

Fig. 9.

If all the active masses lie in finite regions and if the potential of a mass is equal to zero at an infinitely distant reference-point, as in the case of gravitation, then for all infinitely distant reference-points $U = 0$—that is, the infinitely distant spherical surface is an equipotential surface. Then none of the other equipotential surfaces goes off to infinity, but every one of them is a closed surface (Fig. 9) the form of which of course depends on the position of the acting masses.

The representation by equipotential surfaces gives us a direct intuitive knowledge of the characteristic properties of the field of force—that is, of the magnitude and direction of the force ***F*** at any arbitrary reference-point. For if we take the reference-point P (Fig. 9), through which we shall assume the equipotential surface $U = c$ to pass, and describe through it the tangential plane to the surface, then, if dx' denotes an infinitely small displacement of the reference-point in the tangential plane:

$$\frac{dU}{dx'} = 0,$$

and by (120): $X' = 0$—that is, the component of the force $\boldsymbol{F}$ in the direction of any tangent to the equipotential surface is equal to zero, or the force is perpendicular to the tangential plane. Thus if n denotes the normal to the equipotential surface measured in the direction of $\boldsymbol{F}$ the component of $\boldsymbol{F}$ in the direction n is at the same time the total resultant force:

$$F = -\frac{dU}{dn} \quad . \quad . \quad . \quad . \quad . \quad (122)$$

In general, the quantity F has different values at different points of the equipotential surface. The graphical representation of equipotential surfaces also gives us a clear picture of this law. Let us consider two very close equipotential surfaces $U = c$ and $U = c'$, where c' is to be a little smaller than c. Then the force $\boldsymbol{F}$ acts at all points P of the surface c in the direction from c to c', and the magnitude of the force is, by (122):

$$F = \frac{c - c'}{\Delta}$$

where $\Delta = dn$ denotes the (positive) distance in space between the two surfaces. That is, the value of the force is inversely proportional to the distance between the two surfaces. The closer the surfaces lie together, the greater the force.

In this way the equipotential surfaces are closely analogous to the isothermals and isobars represented by curves in meteorological maps; in these maps the place of potential is taken by the temperature or the pressure, the negative gradient of which gives the value and direction of the current of heat-conduction or of the pressure.

The curves which intersect the equipotential surfaces $c, c', c'', \ldots$ normally (indicated by dotted lines in Fig.

9) and in the direction of decreasing potential are called the "lines of force" of the field, since they give the direction of the force which acts at each point of them. For example, in the case of a homogeneous sphere which acts according to the law of gravitation the equipotential surfaces are the concentric spherical surfaces, the lines of force are the straight lines which run from the outside towards the centre. If the sphere has a concentric spherical space in its interior this whole space represents a single degenerate equipotential surface, in which the course of the lines of force remains indeterminate.

In general, the equations of a line of force are, by (110):

$$dx : dy : dz = \frac{\partial U}{\partial x} : \frac{\partial U}{\partial y} : \frac{\partial U}{\partial z} \quad . \quad . \quad . \quad (123)$$

A line of force cannot return into itself, but must either go off to infinity or end at a singular point. For, as it always runs in the direction of decreasing potential, and since, by definition (108), the potential has a single definite value at every point in space except for an arbitrary additive constant, it is impossible for a line of force to return to its starting-point.

§ 41. Let us consider lastly the special case where the reference-point P is in equilibrium—for example, mid-way between two equal attracting point-masses; then by (110):

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0 \quad . \quad . \quad (124)$$

that is, the direction of the line of force which passes through P is indeterminate. A point of equilibrium of this kind, which we shall denote by P_0, is thus a singular point in the system of equipotential surfaces and lines of force. By (124) this is the case, for example, if the function U has an absolute maximum or minimum at P_0. It is easy to see then that in the former case the equilibrium is absolutely unstable and in the latter case absolutely stable. For if the reference-point P is displaced a little from its position of equilibrium P_0, the equations (124) no longer hold, and the point is set into motion by the force which acts on it—namely, in the direction of decreasing potential.

If the potential U is a maximum at P_0 the moving point accordingly cannot return to its position of equilibrium—that is, the equilibrium is unstable. The reverse is true if the potential U has a minimum at P_0.

But the equations (124) may also hold without U being a maximum or a minimum; the answer to the question whether the point when displaced from its position of equilibrium returns to it or not depends on the direction in which the displacement has occurred, and the equilibrium is called conditionally stable or conditionally unstable.

If, finally, U is constant within a finite space, as in the case of the internal space of a hollow sphere discussed in § 38, the equations (124) hold in the whole space. The equilibrium is not disturbed at all then by a displacement of the reference-point and is therefore said to be neutral (*indifferent*).

§ 42. Whereas the above laws, from § 39 onwards, hold for any arbitrary law of attraction, we shall now deal in particular again with Newton's law of gravitation. In the expression for the Newtonian potential U, which is represented by (111) or (112) according as we are dealing with masses distributed as points or spatially, the essential and characteristic feature is the function multiplied by $-fm$, which is therefore often called the "potential function" ϕ in contrast to the potential U. The expression for it is, in the two cases given:

$$\phi = \Sigma \frac{\mu_1}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

and:

$$\phi = \int \frac{k dV}{r} \quad . \quad . \quad . \quad . \quad (126)$$

The most important difference between these two expressions for the potential function is this, that if the reference-point x, y, z moves into one of the active masses ξ, η, ζ the first expression and all its differential coefficients become infinitely great, whereas the second expression, as we saw in § 38, is finite in the interior of the active masses and remains continuous even in passing through the surface.

Let us next also inquire into the differential coefficients of the potential function ϕ in (126) with respect to x, y, z. The first differential coefficients give the components of the attractive force, and hence are, by § 33, finite and continuous throughout. Their values are obtained from (126) by differentiation, if we bear in mind that k depends only on ξ, η, ζ but not on x, y, z:

$$\frac{\partial\phi}{\partial x} = \int\frac{(\xi - x)\,kdV}{r^3} \text{ and so forth}, \quad . \quad . \quad (127)$$

which agrees with (90).

The fact that the quantity $\frac{\partial\phi}{\partial x}$ is finite for an internal point in spite of r^3 in the denominator may be seen directly if we express dV in polar co-ordinates, with the reference-point x, y, z as origin. Then the factor ρ^2 becomes r^2 in the expression (93) for dV, and in (127) we are left, apart from only finite quantities, with only the factor $\frac{\xi - x}{r}$, which is less than 1.

§ 43. The circumstances become different if we pass on to the second differential coefficients of ϕ with respect to x, y, z. For if we again differentiate (127) with respect to x:

$$\frac{\partial^2\phi}{\partial x^2} = -\int\left(\frac{1}{r^3} - \frac{3(x-\xi)^2}{r^5}\right)kdV \quad . \quad . \quad (128)$$

This expression has a definite meaning only if r is different from zero throughout—that is, if the reference-point lies outside all the active masses. For if x, y, z coincides with one of the ξ, η, ζ's, r becomes equal to zero, and by introducing polar co-ordinates we see, as at the end of the preceding section, that every term in 128) tends to infinity logarithmically, so that the value of the difference assumes the indeterminate form $\infty - \infty$.

We therefore first restrict our attention to the case where the reference-point x, y, z lies outside. By forming $\frac{\partial^2\phi}{\partial y^2}$ and $\frac{\partial^2\phi}{\partial z^2}$ analogously and adding these three integrals, we get, taking into account (85), the important relationship, which is characteristic for the Newtonian potential function :

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \equiv \Delta\phi = 0 \,. \qquad (129)$$

which is called *Laplace's Equation.*

§ 44. Let us now inquire into the value of $\Delta\phi$ for a reference-point in the interior of the active masses. The equation (128) is useless for this case; nevertheless $\frac{\partial^2\phi}{\partial x^2}$, and also ϕ and $\frac{\partial\phi}{\partial x}$ have a finite value even in the interior of the masses.

For example, if we take the simple case of a homogeneous sphere of radius R, whose centre is at the origin of coordinates, then we have for a point x, y, z in its interior, by (119) :

$$\phi = \frac{2\pi}{3} k(3R^2 - x^2 - y^2 - z^2) \qquad (130)$$

which is obtained from the expression for the potential U given in (119) if we omit the factor $-fm$ and remember that r_0 represents the distance of the reference-point from the centre of the sphere. Hence it follows that :

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{4\pi}{3} k = \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial z^2}$$

and :

$$\Delta\phi = -4\pi k \qquad (131)$$

which is independent of the radius of the sphere. This last equation is called *Poisson's Equation.*

We may easily generalize Poisson's equation for the case of a non-homogeneous mass of arbitrary shape. For this purpose we imagine a very small sphere described about the reference-point situated in the interior of the mass; we call this very small mass 1 to distinguish it from the remaining mass 2. The potential function ϕ of the whole mass is then equal to the sum of the potential functions due to the mass 1 and those due to the mass 2:

$$\phi = \phi_1 + \phi_2, \text{ and likewise: } \Delta\phi = \Delta\phi_1 + \Delta\phi_2.$$

But by (129) $\Delta\phi_2 = 0$ because the reference-point is an external point with respect to the mass 2, and so we are left with $\Delta\phi = \Delta\phi_1$. Since the sphere is very small we may regard it without appreciable error as being homogeneous and, in fact, as having the density which the active mass has exactly at the point where the reference-point happens to be situated. This is obtained by substituting the values x, y, z for ξ, η, ζ in $k_{\xi,\eta,\zeta}$. Hence it follows by (131) that:

$$\Delta\phi = -4\pi k_{x,y,z} \quad . \quad . \quad . \quad . \quad (132)$$

Poisson's equation may be regarded as a generalization of Laplace's equation in that we may imagine the active masses to occupy the whole of infinite space, without leaving gaps, with a density k, which is partly zero and partly different from zero. The reference-point is then always situated within the masses, and the equation (132) always holds. At places where there is no actual mass $k = 0$, and we get Laplace's equation in place of Poisson's equation. At the same time, we see, if we take this point of view, that the abrupt change which the second differential coefficients of ϕ experience when the reference-point passes through the surface of the masses is due to the sudden change of the density k in this transition.

§ 45. From the mathematical point of view we see that if the potential function ϕ of any spatial distribution of masses is given at all points of space, the density k of these masses can be calculated by a simple unambiguous differential operation, whereas the reverse problem of finding the potential function ϕ from the density k is one involving the integral calculus. In other words, the expression (126):

$$\phi = \int \frac{k_{\xi,\eta,\zeta} \cdot dV}{r}$$

in which we assume k to be infinitesimal at infinitely distant points ξ, η, ζ is an integral of the differential equation (132); it is not the general integral but that particular integral which is limited by the condition that ϕ vanishes when the reference-point moves off to infinity.

The general expression of a single-valued continuous function (whose first differential coefficient is also continuous) which satisfies the differential equation (132) is:

$$\phi = \int \frac{k dV}{r} + \phi_0 \quad . \quad . \quad . \quad . \quad (133)$$

where ϕ_0 satisfies the equation $\Delta\phi_0 = 0$ in the whole of infinite space. ϕ_0 may always be regarded as the potential function of masses which are entirely at infinity. For example, the special value:

$$\phi_0 = \text{const.},$$

which clearly likewise satisfies the equation $\Delta\phi_0 = 0$, is equal to the potential function of a homogeneous spherical layer of infinite radius. (Cf. § 37.)

§ 46. We shall also calculate the potential function for the special case where the density of the active masses is independent of one of the three co-ordinates, say of ζ. This case is realized when the masses are arranged cylindrically parallel to the z-axis in such a way that the density is constant in every infinitely thin cylinder.

The potential function ϕ will then depend only on x and y but not on z, and without loss of generality we may therefore assume the reference-point to lie in the xy-plane: $z = 0$, so that r^2 becomes:

$$r^2 = \zeta^2 + \rho^2.$$

where we have used the abbreviation:

$$\rho^2 = (x - \xi)^2 + (y - \eta)^2 \quad . \quad . \quad . \quad (134)$$

ρ is the distance of the reference-point from the straight line which passes through the point ξ, η, ζ and is parallel to the z-axis. If we now substitute for dV its value given by (88) and use as an abbreviation for the cross-section of an infinitely thin cylinder:

$$d\xi d\eta = d\sigma \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

we get for the required potential function by (126):

$$\phi = \iint \frac{k_{\xi,\eta} d\sigma \, . \, d\zeta}{\sqrt{\zeta^2 + \rho^2}} \quad . \quad . \qquad (136)$$

We first perform the integration with respect to ζ. Then:

$$\phi = \int k d\sigma \cdot \Big[\log(\zeta + \sqrt{\zeta^2 + \rho^2})\Big]_{-l}^{+l}.$$

Here l denotes half the length of an infinitely thin cylinder, which is to be assumed so great that a further increase in size no longer has a physical meaning. If we now consider that for a sufficiently great value of l:

$$\log \frac{l + \sqrt{l^2 + \rho^2}}{-l + \sqrt{l^2 + \rho^2}} = \log \frac{4l^2}{\rho^2} = 2 \log \frac{2l}{\rho},$$

we get:

$$\phi = \int k d\sigma \, . \, 2 \log \frac{2l}{\rho} = \text{const.} - 2 \int k \log \rho \, . \, d\sigma.$$

The numerical value of the constant, although infinitely great, has no physical significance (§ 36). We therefore write the potential function thus:

$$\phi = -2 \int k_{\xi,\eta} \log \rho \, . \, d\sigma \quad . \quad . \quad . \qquad (137)$$

and observe that in this expression everything that refers to the z-direction has vanished, so that it applies exclusively to the *plane*. We may therefore also interpret ϕ as the potential function of certain fictitious masses which are distributed over the xy-plane with a surface-density $2k = \kappa$ in the surface-element $d\sigma$, at a reference-point situated in the same plane and at a distance from

$d\sigma$ which is given by ρ. But the law of force is no longer that of Newtonian attraction.

For we have for the force-components, except for a factor which is of no importance :

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x} &= -\int\frac{\kappa}{\rho}\frac{x-\xi}{\rho}\cdot d\sigma\\ \frac{\partial\phi}{\partial y} &= -\int\frac{\kappa}{\rho}\frac{y-\eta}{\rho}\cdot d\sigma\end{aligned}\right\} \quad . \quad . \quad (138)$$

from which we see that the attractive force is inversely proportional to the distance ρ.

The logarithmic potential :

$$\phi = -\int \kappa_{\xi,\eta}\log\rho\cdot d\sigma, \quad . \quad . \quad . \quad (139)$$

is for the plane what the Newtonian potential is for space. In particular Poisson's equation (132) holds for it :

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = -2\pi\kappa_{x,y} \quad . \quad . \quad . \quad (140)$$

which becomes for a point outside the masses :

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad (141)$$

as we may show directly by differentiating (138) with respect to x and y.

The following theorems also hold for the logarithmic potential; they may be derived from the equations (138) in precisely the same way as the theorems for the attraction of a homogeneous spherical layer according to the Newtonian law may be derived from equations (90).

The attraction of a ring-shaped surface, bounded by two concentric surfaces and with mass distributed uniformly over it, at a point outside it is exactly the same as if the whole mass were concentrated at the centre of the circle; the attraction exerted at a point in the interior of the smaller circle, on the other hand, is zero. Hence we get in exactly the same way as in § 38 the following expression for the potential function of this homogeneous circular ring, if R_1 and R_2 denote the radii of the inner and outer boundary and ρ_0 the distance of the reference-point from the centre :

For $\rho_0 > R_2$:

$$\phi = -\pi\kappa\,(R_2^2 - R_1^2)\log\rho_0 \quad . \quad . \quad (142)$$

For $R_2 > \rho_0 > R_1$:

$$\phi = \frac{\pi\kappa}{2}(R_2^2 - \rho_0^2) - \pi\kappa R_2^2\log R_2 + \pi\kappa R_1^2\log\rho_0 \quad . \quad (143)$$

For $\rho_0 < R_1$:

$$\phi = \frac{\pi\kappa}{2}(R_2^2 - R_1^2) - \pi\kappa R_2^2\log R_2 + \pi\kappa R_1^2\log R_1 \quad . \quad (144)$$

It is easy to show that ϕ and its first differential coefficient with respect to ρ_0 are everywhere continuous, whereas for $\Delta\phi$ the equations (140) and (141) hold.

For a complete circle of radius R we have $R_1 = 0$, $R_2 = R$ and so the potential function for a point at a distance $\rho_0 < R$ from the centre becomes by (143) :

$$\phi = \frac{\pi\kappa}{2}(R^2 - \rho_0^2) - \pi\kappa R^2\log R \quad . \quad (145)$$

whereas for an external point ($\rho_0 > R$) the potential function has, by (142), the value :

$$\phi = -\mu\log\rho_0 \quad . \quad . \quad . \quad . \quad (146)$$

where $\mu = \pi\kappa R^2$ denotes the total attracting mass.

CHAPTER IV

INTEGRATION OF THE EQUATIONS OF MOTION

§ 47. THE problem of determining the motion of a material force under the action of given forces requires the integration of the equations of motion (55) and this again can be performed directly only if the force-components are given either as constants or as functions of the time t. Mostly, however, the force-components will depend on the position of the point or on its velocity, and then the equations of motion require particular treatment if the integration is to be performed. Some of these methods of treatment, which allow the integration to be performed in many cases, will now be described.

If we multiply the equations (55) successively by u, v, w and add, we get:

$$m(u\dot{u} + v\dot{v} + w\dot{w}) = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt},$$

or, by (41), if we multiply by dt:

$$d\left(\frac{1}{2}mq^2\right) = Xdx + Ydy + Zdz \quad . \quad . \quad (147)$$

The quantity $\frac{1}{2}mq^2$ is called, perhaps a little inappropriately, the " vis viva " of the reference-point, whereas the differential expression on the right-hand side is called the " work " A which the force $\boldsymbol{F}$ performs at the reference-point, when it passes from the position x, y, z to the position $x + dx$, $y + dy$, $z + dz$. Using the relationships (60) and (45) we may also express the work in the form:

$$\begin{aligned} A &= F \cdot ds \cdot (\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu) \\ &= F \cdot ds \cdot \cos(\boldsymbol{F}, d\boldsymbol{r}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148) \end{aligned}$$

That is, the work is equal to the product of the magnitude of the force, the magnitude of the displacement and the cosine of the angle between them. If the angle is obtuse the work is negative; if the angle is a right angle the work is zero.

In vector calculus the quantity (148) is called the product of the two vectors $\boldsymbol{F}$ and $d\boldsymbol{r}$:

$$A = \boldsymbol{F} . d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (149)$$

and it is specifically called the "scalar product" because work belongs to the group of scalar quantities (§ 36). The unit of work in the absolute c.g.s. system—that is, the work performed by a force of 1 dyne in displacing the reference-point through 1 cm. in the direction of the force, is called an *erg*.

§ 48. The significance of the equation (147) which states quite generally that the *change in the vis viva of the reference-point is equal to the work performed by the active force* consists in the fact that it enables the integration to be performed directly in numerous important cases. It is true that in general integration is not possible; for even if the force-components X, Y, Z are known as functions of x, y, z, it is not always possible to find a function of x, y, z whose differential is equal to A. For example, if:

$$X = y^2, \ Y = x^2, \ Z = 0$$

then:

$$A = y^2dx + x^2dy$$

In such a case A is called an "incomplete differential" and the integration must be performed in a way different from that used in forming A.

But if the active force is, in particular, a central force which is due to stationary point-masses, then by (110) a potential U exists and the work becomes:

$$A = -dU \quad . \quad . \quad . \quad . \quad . \quad (150)$$

That is, the work of the force is equal to the decrease in U. By substituting in (147) and integrating, we then get:

$$\frac{1}{2}mq^2 - \frac{1}{2}mq_0^2 = U_0 - U \quad . \quad . \quad . \quad (151)$$

where q_0 and U_0 denote the values of q and U at the time $t = 0$. According to this the velocity q depends only on the potential U. Thus if the point passes through a definite equipotential surface $U =$ const., no matter at what point, in what direction or at what time, it always has a definite velocity. From (150) we also derive the physical meaning of the potential U. It is the work which the central force performs as a whole when the reference-point moves in any way from a place where the potential U exists to a place where the potential is zero. The equation (151) is then called the "Principle of Vis Viva."

Actually, we have already applied the principle of vis viva on a number of occasions without characterizing it as such.

For a heavy point of mass m we have:

$$X = 0, \quad Y = 0, \quad Z = -mg.$$

From this we get by (110):

$$U = +mgz + \text{const.} \quad . \quad . \quad . \quad (152)$$

and, if $z = 0$ when $t = 0$, then by (151):

$$\frac{1}{2}mq^2 - \frac{1}{2}mq_0^2 = -mgz \quad . \quad . \quad . \quad (152a)$$

which agrees exactly with (80).

We further have for the example treated in § 12:

$$X = -cx,$$

from which, by (110):

$$U = \frac{1}{2}cx^2 + \text{const.} \quad . \quad . \quad . \quad . \quad (153)$$

and, since $x = 0$ when $t = 0$, by (151):

$$\frac{1}{2}mu^2 - \frac{1}{2}mu_0^2 = -\frac{1}{2}cx^2,$$

which agrees exactly with (17).

When friction comes into play, as in the case treated in § 15, the principle of vis viva no longer applies, because friction is not a central force and therefore the work done by friction does not represent a complete differential.

§ 49. We have seen that the mechanical principle of vis viva is applicable in only a limited region. But as soon as we pass beyond the realm of mechanics it can be formulated in a more general way and expresses a law which is valid without exception in the whole field of physical and chemical phenomena: it is the *Law of Conservation of Energy.* This principle is founded on the recognition of the circumstance which is derived from experimental observations made during several centuries, that it is in no way possible to build a perpetual motion machine—that is, a device by which effects of some kind are continually being produced—without certain other effects being correspondingly used up; or, in other words, that in nature there is a certain quantity E which we may regard as a "capacity" for producing effects and which has the peculiarity that, like the supply of matter that occurs in nature, it can present itself in the most varied forms and is susceptible to various transformations, but its total amount can never be changed, but rather is always conserved: $E = \text{const.}$

The decisive feature in the formulation of the energy principle is the appropriate definition of E, and there have been many differences of opinion and controversies about this point, but not about the validity of the principle itself.

The only way to answer the question correctly is to start out in the first place from particular facts and to seek out those among the relationships which form the expression of these facts that are capable of being interpreted as $E = \text{const.}$ Hence when we look for such a relationship in the realm of the mechanics here being considered we must bear in mind above all that although E may contain the co-ordinates and the velocity components explicitly it must not contain the time t explicitly, because E, being the capacity for producing effects (*Wirkungsvorrat*, stock of effects), can depend only on the instantaneous physical state of the point—that is, on its position and on its velocity. Then there is no longer possibility of doubt. For the only relationship among those found by us which does not contain the time t explicitly is that expressed by the equations (17), (80) and more generally by (151). Hence it follows that the equation (151), which expresses the principle of vis viva :

$$\frac{1}{2}mq^2 + U = \frac{1}{2}mq_0{}^2 + U_0 = \text{const.}, \quad . \quad (154)$$

is to be regarded as the application of the principle of conservation of energy to purely mechanical phenomena, and that in them the mechanical energy is to be written as:

$$E = \frac{1}{2}mq^2 + U = K + U \quad . \quad . \quad (155)$$

Thus the mechanical energy consists of two parts, the “ kinetic ” energy K (energy of motion) and the potential or the “ potential ” energy U (energy of position). Their sum remains constant in all purely mechanical processes.

Since by § 48 the principle of vis viva holds only for such forces as have a potential the mechanical energy remains preserved only for forces of this kind, which are therefore called “ conservative forces.” For non-conservative forces, for example, friction, the mechanical energy changes and the universality of the energy principle demands that in this case the process is not purely mechanical, but that it generates a new kind of energy to an equivalent amount—for example, heat. The equation (154) then becomes generalized in the following way :

$$(K - K_0) + (U - U_0) + W = 0 \quad . \quad . \quad (156)$$

where W denotes the heat generated in the interval of time from 0 to t. For example, the equation (19a), when multiplied by $\frac{dx}{dt}$ and then integrated with respect to t between the limits 0 and t, gives us the relationship :

$$(K - K_0) + (U - U_0) + \rho\int_0^t \left(\frac{dx}{dt}\right)^2 dt = 0 \quad . \quad (157)$$

The last integral represents the heat generated, W.

Another example of a non-conservative force is given by the case where the force is some function of the time t, such as when we act on the material point by means of our muscles according to some arbitrary rhythmical law. We can then, of course, alter the mechanical energy of the point according to our wishes, but the energy principle then demands that the change in mechanical energy be exactly compensated by an equivalent amount of muscular energy.

§ 50. Another method of integrating the equations of motion (55) may always be applied when the direction of the force $\boldsymbol{F}$, no matter what its magnitude, always passes through a fixed centre. Then the orbit of the reference-point lies in a plane which is defined by the centre, the initial position and the initial velocity of the reference-point. Let us choose this plane as the xy-plane and the centre as the origin of co-ordinates, then $z = 0$, $Z = 0$ and :

$$X : Y = x : y$$

Substituted in (55) this gives :

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0 = \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \quad . \quad (157a)$$

and hence by integration :

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c' \quad . \quad . \quad . \quad . \quad . \quad (158)$$

This equation admits of a simple interpretation if we introduce plane polar co-ordinates r and ϕ by means of the relationships :

$$x = r\cos\phi, \quad y = r\sin\phi \quad . \quad . \quad . \quad (159)$$

For then :

$$\left.\begin{aligned} dx &= dr\cos\phi - r\sin\phi d\phi \\ dy &= dr\sin\phi + r\cos\phi d\phi \end{aligned}\right\} \quad . \quad . \quad (160)$$

and the equation (158) becomes :

$$r^2\frac{d\phi}{dt} = c' \quad . \quad . \quad . \quad . \quad . \quad (161)$$

which, expressed as an integral, becomes:

$$\int_{\varphi_0}^{\varphi} r^2 d\phi = c't \quad . \qquad . \qquad (162)$$

Now $r^2d\phi$ is, except for smaller quantities of the second order, twice the area of the infinitely small triangle AOB (Fig. 10) which is formed by the radius vectors OA and OB at the times t and $t + dt$, and the element AB of the orbit. So, according to equation (162), the area which is enclosed by the radius vectors at the times 0 and t and the orbit of the reference-point is proportional to the time t; in other words, the radius vector sweeps out equal areas in equal times. Hence equation (161) or (158) is also called the "Principle of Sectorial Areas."

§ 51. Another application of the law of sectorial areas occurs when the force $\boldsymbol{F}$ is directed, not through a fixed centre, but through a straight line fixed in space. For if we take this line as the z-axis, then we have, not $z = 0$ and $Z = 0$, but $X : Y = x : y$, and equations (161) and (162) follow from this exactly as in § 50. Hence in this case the principle of sectorial areas does not hold for the reference-point itself, but it does hold for the motion of its projection on the xy-plane—that is, for the motion of the point whose co-ordinates are x, y and 0.

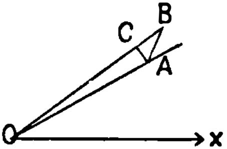

FIG 10.

§ 52. We shall now apply the theorems that have been derived to a special case which is of particular importance in nature, and for this purpose we choose the motion of a material point m which is attracted to a fixed centre μ by a Newtonian gravitational force, just as a planet is attracted by the sun.

We at once see then that the motion occurs in a plane and hence requires only two equations to define it. To obtain these equations we shall use the principle of vis viva and the principle of sectorial areas, and we shall take the plane of the motion as the xy-plane.

Now, by (154), the principle of vis viva gives us, if we substitute the value of the gravitational potential for a single centre from (111) and divide by $\frac{m}{2}$:

$$q^2 - \frac{2f\mu}{r} = c \quad . \quad . \qquad (163)$$

The principle of sectorial areas is given by (161). The values of the constants c and c' are determined by the initial state—namely, if we denote the values for $t = 0$ by adding the suffix 0, we have:

$$c = q_0{}^2 - \frac{2f\mu}{r_0} \quad . \quad . \qquad (164)$$

The constant c' depends, in addition, on the direction of the initial velocity. We find it more expedient to express this direction not by the angle made with the x-axis, because this angle has no physical meaning, but by the angle made with the radius vector, namely α_0; this gives us the advantage that the choice of the x-axis still remains completely open. Now in the right-angled triangle ABC (Fig. 10):

$$AB = ds, \quad AC = rd\phi, \quad \sphericalangle B = \alpha$$

Consequently:

$$rd\phi = ds \, . \sin\alpha$$

and, divided by dt:

$$r\frac{d\phi}{dt} = q\sin\alpha$$

Hence for the initial state, by (161):

$$c' = r_0 q_0 \sin\alpha_0 \quad . \quad . \quad . \quad . \quad . \qquad (165)$$

§ 53. If t is eliminated from the equations of motion (161) and (163) we obtain the orbit of the planet. It is of course advantageous in the present instance to use polar co-ordinates.

Then, by (160):

$$q^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad . \qquad (166)$$

and by (163):

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 - \frac{2f\mu}{r} = c \quad . \quad . \quad . \quad (167)$$

and by eliminating dt from (161) and (167) we get as the differential equation of the orbital curve:

$$d\phi = \frac{c'dr}{r\sqrt{cr^2 + 2f\mu r - c'^2}} \quad . \quad . \quad . \quad (168)$$

Integrating, we get:

$$\phi = \cos^{-1}\frac{c'^2 - f\mu r}{r\sqrt{f^2\mu^2 + cc'^2}} + c'' \qquad . \quad (168a)$$

The integration constant c'' is determined by the value ϕ_0 which ϕ assumes when $r = r_0$. Since we have not yet fixed the direction of the x-axis, we can without loss of generality set $c'' = 0$, which only means that we have decided on our choice of x-axis.

We then obtain:

$$\cos\phi = \frac{c'^2 - f\mu r}{r\sqrt{f^2\mu^2 + cc'^2}} \quad . \quad . \quad . \quad (169)$$

If we solve this equation for r and use the abbreviations:

$$\sqrt{1 + \frac{cc'^2}{f^2\mu^2}} = \epsilon \qquad . \quad . \quad (170)$$

$$\frac{c'^2}{f\mu} = p \quad . \quad . \quad . \quad . \quad (171)$$

we get:

$$r = \frac{p}{1 + \epsilon\cos\phi} \quad . \quad . \quad . \quad . \quad (172)$$

This is the equation to a conic section (Fig. 11) whose focus is at the origin of co-ordinates, whose major axis is along the x-axis and whose latus-rectum (ordinate at the focus) is p and the numerical value of the eccentricity is:

$$\epsilon = \frac{\sqrt{a^2 - b^2}}{a} \quad . \quad . \quad . \quad . \quad (173)$$

The values of the semi-axes are given by :

$$a = \frac{p}{1 - \epsilon^2} \text{ and } b^2 = pa \quad . \quad . \quad . \quad (174)$$

All these constants are obtained from the conditions for the initial state r_0, q_0 and α_0 according to (170) and (171)

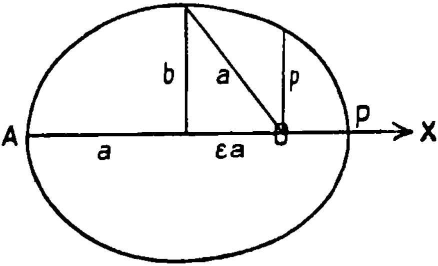

FIG. 11.

if we take into account the values of c and c' in (164) and (165) as follows :

$$\epsilon^2 = \left(\frac{r_0 q_0^2}{f\mu} - 1\right)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0, \; p = \frac{r_0^2 q_0^2 \sin^2 \alpha_0}{f\mu} . \quad (175)$$

$$\frac{1}{\alpha} = \frac{2}{r_0} - \frac{q_0^2}{f\mu}, \; b^2 = \frac{r_0^2 \sin^2 \alpha_0}{\frac{2f\mu}{r_0 q_0^2} - 1} . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

The conic section is an ellipse, parabola or hyperbola according as the eccentricity ϵ is smaller than, equal to, or greater than 1, or, by (170), according as c is negative, zero or positive. Actually, by (163), $\sqrt{c}$ denotes the velocity at an infinite distance, which becomes imaginary for an elliptic orbit. For the initial state it follows from this, by (164), that the conditions for these three kinds of conic section are, respectively :

$$q_0^2 \lesseqgtr \frac{2f\mu}{r_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

It is noteworthy that the species of the conic section and also the length of the semi-major axis a do not depend on the direction of the initial velocity, but only on the sign of the energy constant c.

The elliptic form of the planetary orbits with the sun as a focus constitutes the content of the first of the three laws found empirically by Kepler, the second law is the statement of the principle of sectorial areas, the third follows in the next section.

For the earth's orbit the eccentricity ϵ is approximately equal to $\frac{1}{60}$, the parameter p is approximately equal to $148 \cdot 10^6$ kms., the velocity q averages $30 \frac{\text{kms.}}{\text{secs.}}$, and so the angular velocity is about :

$$\frac{q}{p} = 2 \cdot 10^{-7} \quad . \quad . \quad . \quad . \quad . \quad (178)$$

The condition that the orbit should be circular is : $\epsilon = 0$, and so for the initial state according to (175), since ϵ^2 is the sum of two squares :

$$\cos \alpha_0 = 0 \text{ and } q_0{}^2 = \frac{f\mu}{r_0}.$$

That is, the velocity must in the first place be perpendicular to the radius vector and in the second place must have just that value which leads to the relationship (62) which holds for uniform circular motion. The latter condition becomes clear at once if we remember that here the force $F = f \frac{m\mu}{r^2}$.

§ 54. More complicated relationships hold for the dependence of the co-ordinates r and ϕ on the time t, and their evaluation in problems of astronomy can be performed only by means of expansion in series. Assuming that the path is elliptic, we shall now also calculate the time T required for one revolution. This can be done most simply by using equation (162) of the principle of sectorial areas and performing the integration from $t = 0$ to $t = T$—that is, from $\phi = \phi_0$ to $\phi = \phi_0 + 2\pi$. On the left-hand side we then get the double area of the ellipse, that is :

$$2ab\pi = c'T.$$

If we eliminate c' by means of (171) and b by means of (174) we get :

$$T^2 = \frac{4\pi^2 a^3}{f\mu} \quad . \quad . \quad . \quad . \quad . \qquad (179)$$

That is, for a given μ (the sun) the square of the time of revolution is proportional to the cube of the major axis (Kepler's third law).

In this way the three Kepler laws which in their original form appeared to be in no way inherently related present themselves as deductions from the one Newtonian law of gravitation. But the significance of Newton's law does not alone consist in the fact that it allows Kepler's laws to be derived from it. It embraces above all the laws which govern weight on the earth. For we may also apply equation (179) to the earth as the centre of attraction with its mass μ by substituting for T, the time of revolution of the moon, and for a, the radius of the moon's orbit. But by (103) the value for $f\mu$ which then follows is equal to $R^2 g$ where R denotes the radius of the earth and g the acceleration due to gravity at the earth's surface. Thus :

$$g = \frac{4\pi^2 a^3}{R^2 T^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (180)$$

If we substitute in this expression the values :

$$a = 60{\cdot}1 \,.\, R$$
$$R = 637 \,.\, 10^6 \text{ cms.}$$
$$T = 1 \text{ month} = 236 \,.\, 10^4 \text{ secs.}$$

we get $g = 981$, which agrees sufficiently well with measurements on the earth (§ 10). This calculation gave Newton the firm foundation for his theory of general gravitation.

But Newton's law of gravitation does far more than combining the earth's gravitation and Kepler's laws in a single expression. For as was found subsequently it also gives the perturbations caused by the mutual gravitation of the planets as well as a number of other celestial phenomena in complete agreement with the results of observation (motions of comets, double stars and so forth). It is not only simpler but also more accurate than Kepler's laws. Since the efficiency of such a hypothesis can certainly not be ascribed to chance, it seems justifiable to conclude that the enunciation of Newton's law of gravitation is not *au fond* an expedient invention, as some physicists maintain, but rather is to be regarded as an epistemological discovery.

CHAPTER V

RELATIVE MOTION

§ 55. In deriving the laws governing the motions of planets in the preceding chapter we assumed the sun to be at rest. This is, however, certainly not true, since the sun can move freely and is attracted by every planet according to the law of gravitation. To be accurate, then, we must also take this motion into account.

Besides this, however, there is also another circumstance to be considered. What we observe and measure, and therefore what we may alone use in any test of the theory, is not the absolute motion of the sun and the planets at all, but rather the motion such as it appears to us inhabitants of the earth. For we have not a stationary co-ordinate system at our disposal, but rather the co-ordinate system to which we refer the motions of all bodies, including celestial bodies; this system moves with the earth around the sun and, indeed, turns in various directions during the course of the day.

We shall therefore attack the problem at once from the widest angle and shall inquire into the laws of motion of a material point, such as they appear to an observer moving in a definite way.

For this purpose we imagine that besides having the stationary co-ordinate system x, y, z which we have hitherto used (we may leave the possibility of its realization out of the question altogether), we also have another rectangular co-ordinate system x', y', z', which moves in a manner definitely given, and we suppose that at the origin O' of this co-ordinate system we have an observer B' rigidly connected with O', say in such a way that for him the z'-axis points upwards, the x'-axis to the right and consequently the y'-axis straight in front of him.

The question now is: What equations of mechanics hold for the observer B' instead of the equations (55) or (57)? We obtain the answer to this question by expressing the components of the acceleration $\ddot{\boldsymbol{r}}$ on the one hand and the components of the force $\boldsymbol{F}$ on the other hand in terms of the components of the corresponding quantities $\ddot{\boldsymbol{r}}'$ and $\boldsymbol{F}'$ referred to the moving observer, and substituting these values in (55).

§ 56. The problem of expressing $\ddot{\boldsymbol{r}}'$ in terms of $\boldsymbol{r}$ or conversely is purely one of kinematics. Since the motion of the accented co-ordinate system is regarded as known, the co-ordinates x_0, y_0, z_0 of the origin O' as well as the direction-cosines of the three axes x', y', z' are known functions of the time t. We call them α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3, where we allocate the letters α, β, γ to the unaccented axes x, y, z and the numbers 1, 2, 3 to the accented axes x', y', z'. Generalizing the result of (36) we then get:

$$\left.\begin{aligned} x' &= \alpha_1(x - x_0) + \beta_1(y - y_0) + \gamma_1(z - z_0) \\ y' &= \alpha_2(x - x_0) + \beta_2(y - y_0) + \gamma_2(z - z_0) \\ z' &= \alpha_3(x - x_0) + \beta_3(y - y_0) + \gamma_3(z - z_0) \end{aligned}\right\} \quad . \quad . \quad (181)$$

To these three equations there must be added three exactly corresponding equations which are obtained by considering that the direction-cosines of the three unaccented axes x, y, z with respect to the accented axes x', y', z' are, successively: α_1, α_2, α_3, β_1, β_2, β_3 and γ_1, γ_2, γ_3: hence:

$$\left.\begin{aligned} x - x_0 &= \alpha_1 x' + \alpha_2 y' + \alpha_3 z' \\ y - y_0 &= \beta_1 x' + \beta_2 y' + \beta_3 z' \\ z - z_0 &= \gamma_1 x' + \gamma_2 y' + \gamma_3 z' \end{aligned}\right\} \quad . \quad . \quad . \quad (182)$$

To find the relationships between the velocity-components u', v', w' of a material point, such as they appear to the observer B', and the components u, v, w referred to the stationary system, we need only differentiate the equations (181) or (182) with respect to the time t, remembering, however, that the direction-cosines in general depend on the time t.

A second differentiation with respect to the time t gives us the relationships between the acceleration-components $\dot{u}'$, $\dot{v}'$, $\dot{w}'$ on the one hand and $\dot{u}$, $\dot{v}$, $\dot{w}$ on the other, and this completely solves the problem of expressing the vector $\ddot{\boldsymbol{r}}'$ in terms of the vector $\ddot{\boldsymbol{r}}$.

§ 57. From the physical point of view it is more difficult to define the important force-vector $\boldsymbol{F}'$ for the observer B'. To do this two ways immediately present themselves, both of which represent a generalization of equation (56). We might think either of setting the force $\boldsymbol{F}'$ generally equal to mass times acceleration $\dot{\boldsymbol{q}}$, and hence also X equal to $m\dot{u}'$; or we might generally put X' equal to $\alpha_1 X + \beta_1 Y + \gamma_1 Z$. These two definitions are mutually contradictory, since in general $m\dot{u}'$ differs from $\alpha_1 X + \beta_1 Y + \gamma_1 Z$, as is easily seen if we differentiate the first of the equations (181) with respect to t and then compare the values (55).

We can come to a decision as to which of these alternatives is to be adopted only by reverting to the fundamental train of ideas described in § 8 in setting up the concept of force, according to which force is to be regarded from the very beginning not as an acceleration but as the objective cause of motion. For if in general the observer B' were to set force equal to mass times acceleration he would be compelled to conclude that if the force $\boldsymbol{F}'$ which acts on a material point is equal to zero, the velocity of the point $\boldsymbol{q}'$ is constant in magnitude and direction.

Now there is *one* case where the force is undoubtedly zero for *every* observer—namely, the case where the material point is completely isolated and is situated in empty space at an infinite distance from all other bodies (§ 7). Then there is no cause for motion in the objective sense, and hence no force, so that $\boldsymbol{F}'$ is equal to zero. But in this case the velocity of the point for the observer will by no means be constant in magnitude and direction, as the simplest experiment shows, but will depend entirely on how the observer moves, whether, for example, he rotates. Hence $\dot{\boldsymbol{q}}'$ does not in general vanish with $\boldsymbol{F}'$ and the definition of $\boldsymbol{F}'$ on which we are fixing our attention is untenable. On the other hand, the reflection just made shows that $\boldsymbol{F}'$ always vanishes simultaneously with $\boldsymbol{F}$ and this condition leads us to adopt the second of the above alternatives for the definition of $\boldsymbol{F}'$, so that we set generally :

$$\left.\begin{aligned} X' &= \alpha_1 X + \beta_1 Y + \gamma_1 Z \\ Y' &= \alpha_2 X + \beta_2 Y + \gamma_2 Z \\ Z' &= \alpha_3 X + \beta_3 Y + \gamma_3 Z \end{aligned}\right\} \quad . \quad . \quad . \tag{183}$$

and conversely :

$$\left.\begin{aligned} X &= \alpha_1 X' + \alpha_2 Y' + \alpha_3 Z' \\ Y &= \beta_1 X' + \beta_2 Y' + \beta_3 Z' \\ Z &= \gamma_1 X' + \gamma_2 Y' + \gamma_3 Z' \end{aligned}\right\} . \quad . \quad . \tag{184}$$

In principle this solves the problem of relative motion. For if on the one hand we take the components of the force $\boldsymbol{F}$ from (184) and on the other hand the components of the acceleration $\dot{\boldsymbol{q}}$ from (182) and substitute these values in (55) we get the relationships between $\boldsymbol{F}'$ and $\dot{\boldsymbol{q}}'$.

In performing this calculation we shall, however, restrict ourselves for the sake of simplicity to a few special cases of particular importance.

§ 58. The simplest case is that where the accented co-ordinate system is immovably fixed to the unaccented co-ordinate system—that is, where both x_0, y_0, z_0 and also the nine direction-cosines are independent of the time *t*.

If we differentiate (181) and (182) we then get the simple relationships :

$$u' = \alpha_1 u + \beta_1 v + \gamma_1 w, \quad . \quad . \quad . \quad . \quad . \quad . \tag{185}$$

$$u = \alpha_1 u' + \alpha_2 v' + \alpha_3 w', \quad . \quad . \quad . \quad . \quad . \tag{186}$$

$${\dot{}}' = \alpha_1 \dot{u} + \beta_1 \dot{v} + \gamma_1 w, \quad . \quad . \quad . \quad . \quad . \quad . \tag{187}$$

$$\dot{u} = \alpha_1 \dot{u}' + \alpha_2 \dot{v}' + \alpha_3 \dot{w}', \quad . \quad . \quad . \quad . \quad . \tag{188}$$

from which, in view of (183) and (55), it follows that :

$$X' = m\dot{u}', \; Y' = m\dot{v}', \; Z' = m\dot{w}' \quad . \quad . \tag{189}$$

that is, the same equations of motion hold for the accented system of reference as for the unaccented system, or: the equations of motion are "invariant" with respect to the transformation of co-ordinates that has been effected. The velocity and the acceleration also retain their value, whereas the components alter.

§ 59. Let the origin of co-ordinates O' of the accented co-ordinate system move arbitrarily, but let the directions of the axes x', y', z' always remain parallel to the axes x, y, z.

Then x_0, y_0, z_0 are dependent on the time, whereas:

$$\left.\begin{aligned}\alpha_1 = 1,\ \beta_1 = 0,\ \gamma_1 = 0\\ \alpha_2 = 0,\ \beta_2 = 1,\ \gamma_2 = 0\\ \alpha_3 = 0,\ \beta_3 = 0,\ \gamma_3 = 1\end{aligned}\right\} \quad . \quad . \quad . \quad (190)$$

In this case the equations (181) become:

$$\left.\begin{aligned}x' &= x - x_0, \ . \ . \ .\\ u' &= u - u_0, \ . \ . \ .\\ \dot{u}' &= \dot{u} - \dot{u}_0, \ . \ . \ .\end{aligned}\right\} \quad . \quad . \quad (191)$$

and the equations (183) become:

$$X' = X,\ Y' = Y,\ Z' = Z \quad . \quad . \quad . \quad (192)$$

so that by (55) we get as the resulting equations of motion:

$$m\dot{u}' = X' - m\dot{u}_0, \ . \ . \ . \quad . \quad . \quad . \quad (193)$$

which differ from the equations (55) for a stationary system of reference. Since the equations of motion can be tested experimentally, the observer B' has a means of gaining information about his motion with respect to a stationary co-ordinate system by mechanical means. He can, however, measure only the acceleration components $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$, but not the velocity components u_0, v_0, w_0.

We see actually that if we assume the accented system to be in *uniform* motion, say so that:

$$x' = x - u_0t,\ y' = y - v_0t,\ z' = z - w_0t \quad . \quad (194)$$

where u_0, v_0, w_0 are constant, then the equations (189) again result from (193)—that is, the equations of mechanics are invariant also with respect to the transformation (194), which is called a " Galileo transformation " in honour of the discoverer of the law of inertia. Thus an observer B' who is moving uniformly can never find out anything about the velocity of his motion by mechanical measurements; nor are we able to specify a point in world-space about which we can assert that it is absolutely at rest. Rather, an additive constant which is undefined and undefinable remains in *every* velocity. This law is called the classical "Principle of Relativity." (It must be carefully distinguished from the modern Principle of Relativity of Einstein, which correlates from a higher point of view the invariance, discussed in the preceding section, of the equations of motion with respect to a rotation of the co-ordinate system and the invariance with respect to a uniform translation of the origin of co-ordinates.)

It is noteworthy that besides the velocity also the kinetic energy of a material point may be defined only relatively; and, moreover, since the kinetic energy depends quadratically on the velocity, there is left undetermined in the expression for the kinetic energy not only an additive constant, but also a linear function of the velocity. From this we see how necessary it is in all our calculations with mechanical quantities to characterize accurately the system of reference which is being used. As soon as this is done all indefiniteness, of course, vanishes.

§ 60. We shall make a further application of equations (193) to planetary motion by first regarding the sun as freely movable, as described in § 55, its co-ordinates being x_0, y_0, z_0, and investigating the motion of the planet with respect to an observer B' situated on the sun. Let the directions of the accented co-ordinate system be parallel to those of the unaccented system. Although x_0, y_0, z_0 are not given here as functions of the time, yet we can easily find these quantities with the help of the equations of motion for the sun. For by the principle of action and reaction (Newton's third law) the force of attraction exerted by the planet m on the sun μ is equal and opposite to the force X, Y, Z which the sun exerts on the planet; thus:

$$\mu \dot{u}_0 = -X, \ldots$$

Substituted in (193) this gives:

$$m\dot{u}' = X' + X \cdot \frac{m}{\mu}, \ldots$$

and by (192):

$$m\dot{u}' = X \cdot \frac{\mu + m}{\mu}, \ldots \qquad (195)$$

Now by (84) and (191):

$$X = -f \cdot \frac{m\mu}{r^2} \cdot \frac{x'}{r}, \ldots$$

where:

$$r^2 = x'^2 + y'^2 + z'^2.$$

Consequently the equations for the relative motion run:

$$m\dot{u}' = -f \cdot \frac{m(\mu + m)}{r^2} \cdot \frac{x'}{r}, \ldots \qquad (196)$$

This is the motion of a material point of mass m, which is attracted according to the law of gravitation by a centre which is *at rest* at the origin of co-ordinates and which has a mass $m + \mu$. This theorem refers the laws governing the relative motions of planets to those derived in §§ 52 to 54 for absolute planetary motion. The fact that the factor μ is replaced by the greater factor $\mu + m$, which makes the attractive force appear greater than it really is, arises from the circumstance, of course, that the planet draws somewhat nearer to the sun when the sun is freely movable than when it is fixed.

But we may go a step still further. There is nothing to prevent our following the same argument and making the same calculation for the motion of the sun relative to an observer situated on the planet. For we have introduced no restrictions about the ratio of the magnitude of the quantities μ and m.

We may therefore enunciate the following theorem:

for an observer situated on the planet the sun moves in an ellipse (the ecliptic), at one focus of which the planet is situated, according to the principle of sectorial areas, exactly as if the planet were fixed and had the mass $\mu + m$. This ellipse is of course exactly the same as that of the planetary orbit relative to the sun.

§ 61. To approximate more closely to the conditions which obtain on the earth—and we are, after all, dependent on them alone—we now investigate the equations of motion of a material point for a co-ordinate system which rotates with a constant positive angular velocity ω. We first make the origin O' of the rotating system coincide with the origin O, and the axis of rotation z' coincide with the z-axis of the stationary system. If ϕ is then the angle which the x'-axis makes with the x-axis, then we have (Fig. 12):

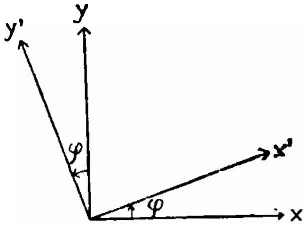

FIG. 12.

$$\left.\begin{array}{lll}\alpha_1 = \cos\phi & \beta_1 = \sin\phi & \gamma_1 = 0 \\ \alpha_2 = -\sin\phi & \beta_2 = \cos\phi & \gamma_2 = 0 \\ \alpha_3 = 0 & \beta_3 = 0 & \gamma_3 = 1\end{array}\right\} \quad (197)$$

where we may set:

$$\phi = \omega t \quad . \quad . \quad . \quad . \quad (198)$$

We then obtain from the equations (181):

$$\left.\begin{array}{l}x' = x\cos\phi + y\sin\phi \\ y' = -x\sin\phi + y\cos\phi \\ z' = z\end{array}\right\} \quad . \quad . \quad (199)$$

and by differentiating:

$$\left.\begin{array}{l}u' = u\cos\phi + v\sin\phi + \omega y' \\ v' = -u\sin\phi + v\cos\phi - \omega x' \\ w' = w\end{array}\right\} \quad . \quad . \quad (200)$$

Similarly:

$$\left.\begin{aligned} \dot{u}' &= \dot{u}\cos\phi + \dot{v}\sin\phi + 2\omega v' + \omega^2 x' \\ \dot{v}' &= -\dot{u}\sin\phi + \dot{v}\cos\phi - 2\omega u' + \omega^2 y' \\ \dot{w}' &= \dot{w} \end{aligned}\right\} \quad (201)$$

If we multiply these equations by m, then if we take into account (55) and (183) we obtain the required differential equations in which, since only accented quantities now occur, we may omit all the accents:

$$\left.\begin{aligned} m\dot{u} &= X + 2m\omega v + m\omega^2 x \\ m\dot{v} &= Y - 2m\omega u + m\omega^2 y \\ m\dot{w} &= Z \end{aligned}\right\} \quad . \quad . \quad (202)$$

Thus the laws of mechanics undergo a change for the rotating observer; this change may be characterized by saying that the "true" or "objective" force, whose components are X, Y, Z, and which may be so produced by the muscles, say (§ 8), is supplemented by two "apparent" or "subjective" forces whose components are $m\omega^2 x$, $m\omega^2 y$, 0 and $2m\omega v$, $-2m\omega u$, 0. The *first* additional force, which depends only on the *position* of the reference-point, is, by § 25, equal in magnitude and exactly opposite in direction to the centripetal force which occurs when the motion of the reference-point is one of rotation about the axis of rotation with the angular velocity ω: it is therefore called the "centrifugal force." The second additional force which depends only on the *velocity* of the reference-point in the moving system is called the "Coriolis force"; it is perpendicular both to the axis of rotation z and to the velocity $\mathbf{q}$, as we see by adding together its components when they have been multiplied respectively by u, v, w. This Coriolis force forms with $\mathbf{q}$ and z a right-handed system (§ 16), which is in general, of course, *not* rectangular. That is, if the observer, for whom the axis of rotation z is directed upwards, looks in the direction $\mathbf{q}$, the Coriolis force acts towards his right-hand side. The magnitude of this force is twice the product of the mass m of the reference-point, of the angular velocity ω and of that component of the velocity q which is perpendicular to the axis of rotation.

§ 62. We now displace the origin of the moving system from the point O of the axis of rotation (centre of the earth) to a point O' on a spherical surface (earth's surface) whose radius is R (earth's radius), and this spherical surface is to rotate together with the system above considered. For simplicity we assume O' to lie in the x, z-plane, the picture plane of Fig. 13. We take the z'-axis in the direction of the earth's radius pointing outwards, the y'-axis parallel to the y-axis (downwards in the figure). Then the x'-axis lies in the plane of the diagram. By transforming to the new accented system we then get the equations of mechanics for an observer B' who is standing on the earth's surface in such a way that for him the z'-axis points upwards, the y'-axis eastwards and the x'-axis southwards.

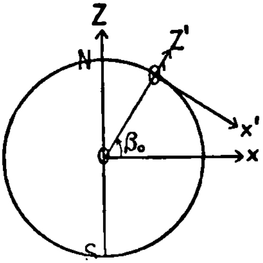

Fig. 13.

Since the new system is rigidly connected with the former system, the simple relationships (185)–(188) hold for this transformation. Now if β_0 denotes the angle which the earth's radius OO' makes with the x-axis (the equator), positive for the northern hemisphere and negative for the southern, then the co-ordinates of O' in the former system are:

$$x_0 = R\cos\beta_0,\ \ y_0 = 0,\ \ z_0 = R\sin\beta_0\ . \qquad . \qquad (203)$$

Further, the direction-cosines of the accented axes with respect to the unaccented axes are:

$$\left.\begin{array}{lll} \alpha_1 = \sin\beta_0, & \beta_1 = 0, & \gamma_1 = -\cos\beta_0 \\ \alpha_2 = 0, & \beta_2 = 1, & \gamma_2 = 0 \\ \alpha_3 = \cos\beta_0, & \beta_3 = 0, & \gamma_3 = \sin\beta_0 \end{array}\right\} . \qquad (204)$$

Consequently, by (187):

$$\begin{aligned} \dot{u}' &= \dot{u} \sin \beta_0 - \dot{w} \cos \beta_0, \\ \dot{v}' &= \dot{v}, \\ \dot{w}' &= \dot{u} \cos \beta_0 + \dot{w} \sin \beta_0. \end{aligned}$$

We multiply these equations by m and substitute for $\dot{u}$, $\dot{v}$, $\dot{w}$ their values from the equations of motion (202). Finally we express all unaccented quantities in terms of accented quantities, namely by (182):

$$\begin{aligned} x &= R \cos \beta_0 + x' \sin \beta_0 + z' \cos \beta_0, \\ y &= y', \end{aligned}$$

and by (186):

$$\begin{aligned} u &= u' \sin \beta_0 + w' \cos \beta_0, \\ v &= v', \end{aligned}$$

and by (184):

$$\begin{aligned} X &= X' \sin \beta_0 + Z' \cos \beta_0, \\ Y &= Y', \\ Z &= - X' \cos \beta_0 + Z' \sin \beta_0. \end{aligned}$$

If we now omit all the accents simultaneously we obtain the equations of motion for an observer standing on the earth's surface in the direction of the earth's radius in the following form, no terms being neglected:

$$\left.\begin{array}{l} \text{Southwards}: m\dot{u} = \\ X + 2m\omega v \sin\beta_0 + m\omega^2 \sin\beta_0 (R\cos\beta_0 + x\sin\beta_0 + z\cos\beta_0) \\ \text{Eastwards}: m\dot{v} = \\ Y - 2m\omega\,(u \sin\beta_0 + w\cos\beta_0) + m\omega^2 y \\ \text{Upwards}: m\omega = \\ Z + 2m\omega v\cos\beta_0 + m\omega^2\cos\beta_0 (R\cos\beta_0 + x\sin\beta_0 + z\cos\beta_0) \end{array}\right\} (205)$$

If the distance of the reference-point from the location of the observer is small compared with the earth's radius, we may neglect the terms in x, y, z as being much smaller than those in R and we get the simpler equations :

$$\left.\begin{aligned}&\text{Southwards:}\\ &\quad m\dot{u} = X + 2m\omega v \sin\beta_0 + m\omega^2 R \sin\beta_0 \cos\beta_0\\ &\text{Eastwards:}\\ &\quad m\dot{v} = Y - 2m\omega(u \sin\beta_0 + w \cos\beta_0)\\ &\text{Upwards:}\\ &\quad m\dot{w} = Z + 2m\omega v \cos\beta_0 + m\omega^2 R \cos^2\beta_0\end{aligned}\right\} \quad . \quad (206)$$

§ 63. Let us next consider the case where the reference-point is subject only to its own weight. Then the force acting on it is, $X = 0$, $Y = 0$, $Z = -mg_0$, where g_0, the acceleration due to gravity for an observer who does not move with the earth, has the value $\frac{fm}{R^2}$, by (103). Hence if we allow the point to fall with the (relative) initial velocity zero—that is, from " rest," then, so long as the velocity u, v, w is still very small, the following relationships hold :

$$\left.\begin{aligned}\dot{u} &= \omega^2 R \sin\beta_0 \cos\beta_0\\ \dot{v} &= 0\\ \dot{w} &= -g_0 + \omega^2 R \cos^2\beta_0\end{aligned}\right\} \quad . \quad . \quad (207)$$

Thus the acceleration is constant, but its value and direction differ from those of g_0. The square of the acceleration is :

$$g^2 = \dot{u}^2 + \dot{w}^2 = g_0{}^2 - 2g_0\omega^2 R \cos^2\beta_0 + \omega^4 R^2 \cos^2\beta_0.$$

Here the third term plays no appreciable part, since the ratio $\frac{\omega^2 R}{g_0}$ for :

$$\omega = \frac{2\pi}{24\,.\,60\,.\,60} = 7{\cdot}272\,.\,10^{-5}\ \text{sec.}^{-1},$$

$$R = 637\,.\,10^6\ \text{cm.},$$

$g_0 = 983\,\frac{\text{cm.}}{\text{sec.}^2}\left(\text{acceleration measured at the pole, for } \beta_0 = \frac{\pi}{2}\right)$

has the value 0·00343. Hence the acceleration itself comes out very nearly as :

$$g = g_0 - \omega^2 R \cos^2\beta_0 = 983 - 3{\cdot}37 \cos^2\beta_0 \quad . \quad (208)$$

whereas we get from the most accurate measurements with a pendulum:

$$g = 983 - 5{\cdot}2\cos^2\beta_0 \quad . \quad . \quad . \quad . \qquad (209)$$

The fact that the decrease of the acceleration due to gravity is considerably greater when we approach the equator than according to the theory here developed is due to the circumstance that the earth is not exactly spherical in shape, but is flattened.

The direction of the acceleration due to gravity—that is, the direction of the plumb-line or of the vertical, by means of which the zenith-point of the observer is determined—does not, by (207), coincide with the z-axis on the earth's radius, but rather has a component which is perpendicular to it. The angle δ made with the earth's radius, since it is very small, amounts to:

$$\delta = \frac{\omega^2 R \sin\beta_0 \cos\beta_0}{g_0} = \frac{\omega^2 R \sin 2\beta_0}{2g_0}\,. \qquad (210)$$

For the pole and the equator $\delta = 0$. On the northern hemisphere δ is positive, on the southern it is negative—that is, on the former the plumb-line is deflected southwards from the direction pointing to the earth's centre, on the latter it is deflected towards the north. The maximum value attained by δ is for $\beta_0 = \frac{\pi}{4}$, namely:

$$\delta_{\text{max.}} = \frac{\omega^2 R}{2g_0} = 0{\cdot}00171 = 5{\cdot}9 \text{ minutes} \quad . \qquad (211)$$

The vertical also determines the geographical latitude β of the observer; it is the angle between the vertical and the equator.

Hence the following relationship holds for this angle:

$$\beta - \beta_0 \doteq \delta \quad . \quad . \quad . \quad . \quad . \quad . \qquad (212)$$

§ 64. The equations of motion of a heavy point-mass become a little simpler if we choose as the z-axis, not the earth's radius, but the vertical—that is, if in the table (204) giving the direction-cosines we replace the angle β_0 by the geographical latitude β. For in that case the southward component of the acceleration in equations (207) drops out, and the equations (206) give in this case:

$$\left.\begin{aligned}&\text{Southwards: } \dot{u} = 2\omega v \sin\beta\\ &\text{Eastwards: } \dot{v} = -2\omega(u \sin\beta + w\cos\beta)\\ &\text{Upwards towards the zenith: } \dot{w} = -g + 2\omega v \cos\beta\end{aligned}\right\} \quad (213)$$

Let us now also follow the motion of the point-mass for greater velocities by letting it drop with an initial velocity zero from a high tower of height h. In this case, too, we may simplify the discussion considerably by taking advantage of the peculiarities here involved.

Of the three acceleration components $\dot{w}$ is of a higher order of magnitude than $\dot{u}$ and $\dot{v}$; hence we may also neglect u and v in comparison with w, and we get, more simply:

$$\begin{aligned}\dot{u} &= 0,\\ \dot{v} &= -2\omega w \cos\beta = -2\omega\cos\beta\cdot\frac{dz}{dt},\\ \dot{w} &= -g.\end{aligned}$$

With the initial conditions ($t = 0$):

$$x = 0,\ y = 0,\ z = h,$$
$$u = 0,\ v = 0,\ w = 0,$$

the first and the third equations give on being integrated:

$$u = 0, \qquad x = 0,$$
$$w = -gt,\ z = -\frac{1}{2}gt^2 + h,$$

but the second gives:

$$v = -2\omega\cos\beta\,.\,(z - h) = \omega g\cos\beta\,.\,t^2,$$
$$y = \frac{1}{3}\omega g\cos\beta\,.\,t^3.$$

Hence by eliminating t we get as the path of the freely falling point:

$$y = \frac{\omega}{3}\cos\beta\,.\sqrt{\frac{8(h-z)^3}{g}} \quad . \quad . \quad . \quad (214)$$

which is called a Neil parabola (Fig. 14); it lies in the vertical plane which points eastward.

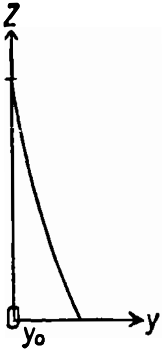

FIG. 14.

The deviation from the vertical which passes through the apex of the tower ($z = h$) amounts at its foot ($z = 0$) to:

$$y_0 = \frac{\omega}{3} \cos\beta \, . \sqrt{\frac{8h^3}{g}}.$$

Thus for $\beta = \frac{\pi}{4}$ and $h = 10^4$ cms. it is:

$$y_0 = 1{\cdot}5 \text{ cms.},$$

which agrees with the results of numerous measurements.

If besides gravity another force, whose components are X, Y, Z, acts on the point-mass, the equations of motion (213) become generalized to:

$$\left.\begin{array}{ll} \text{Southwards:} & m\dot{u} = X + 2m\omega v \sin\beta \\ \text{Eastwards:} & m\dot{v} = Y - 2m\omega(u \sin\beta + w \cos\beta) \\ \text{Upwards towards the zenith:} & \\ & m\dot{w} = Z - mg + 2m\omega v \cos\beta \end{array}\right\} . \qquad (215)$$

CHAPTER VI

CONSTRAINTS

§ 65. HITHERTO we have assumed that the material reference-point that has been under consideration was subject to no influences other than certain forces each of which tended to set it into motion according to a definite law which was assumed known. There are cases, however, where the motion of the point is influenced by causes other than the forces given at the beginning, as, for example, when the point is compelled to remain on a fixed surface or on a fixed curve, or, more generally, when the motion of the point is subject to certain conditions prescribed from the outset. The question is: according to what rules are these cases to be treated?

To solve this problem we again revert to our original derivation of the force-concept. If we maintain the view that *every* causal influence that affects the motion of the point always makes itself felt through a certain force, we must conclude that even a prescribed condition can become physically effective only if it can be realized through a certain force. If we add this force to the other given forces the point moves exactly like the so-called "free" point that we have hitherto been considering. It is true that the forces of this new kind have properties essentially different from those so far treated, as may easily be recognized by the fact that their magnitude is not directly given, but depends on the remaining forces.

Hence we shall in future call a force of this kind a "constraining force," **Z** (*Zwangskraft*), to distinguish it clearly from the "driving or propelling forces," **F** (*treibende Kräfte*), which have so far been alone considered and which we shall still continue to regard as being given.

After what has been said, the equations of motion (57) then become generalized to:

$$m\dot{\boldsymbol{q}} = \boldsymbol{F} + \boldsymbol{Z} \quad . \quad . \quad . \quad . \quad . \qquad (216)$$

where $\boldsymbol{F}$ denotes the resultant of all the driving forces and $\boldsymbol{Z}$ the resultant of all the constraining forces. The total resultant $\boldsymbol{F} + \boldsymbol{Z}$ is also called the "moving force" or, more commonly, the "effective force."

It is clear that the equations (216) are not sufficient to determine the motion; for there are now three new unknowns in it, namely the components of $\boldsymbol{Z}$. Hence we require three further equations and must therefore look round for further conditions. In the first place we find the prescribed conditions themselves, of which we shall assume that they may be represented by one or more equations between the co-ordinates x, y, z of the reference-point. A single equation, alone, means that the point is constrained to remain on a given surface; two equations denote that the point can move only on a given curve. This exhausts all the possibilities that come into question; for when there are three equations the point is fixed and hence its position is given for all time.

But the prescribed conditions are not yet sufficient in themselves; rather, we require still other properties of the constraining forces $\boldsymbol{Z}$. These can be found only if we consider the prescribed conditions to be realized materially in some way. For example, if the reference-point is constrained to remain on a fixed curve, we imagine it to be movable in a very narrow fixed tube or, say, to be punctured to allow a suitably bent but very strong wire to pass through it, so that the point can move along the wire: in each case the motion must of course occur without friction, because the constraining force only prevents the point from leaving the curve, whereas it is of no significance for the motion along the curve. It follows immediately from these considerations that the constraining force can have no component in the direction of the tangent to the curve, neither an accelerating nor a retarding component—

in other words, the constraining force must be in a direction normal to the curve. In the same way, we must conclude that the constraining force due to a fixed surface must always act normally to the curve.

It is easy to see that this law which governs the direction of the constraining force, when taken in conjunction with the prescribed conditions, completes the three equations which we found to be necessary above for supplementing the general equations of motion (216) if we are to be able to find the motion of the reference-point as well as the magnitude and direction of the constraining force. For in the case of a fixed curve we have *two* prescribed conditions and we have as our third equation that which asserts that the constraining force acts in a direction normal to the curve. In the case of a fixed surface, we have only one prescribed condition, but in addition we have the two equations which express that the constraining force coincides with the normal to the surface—that is, that it lies in a perfectly definite direction. For the sake of completeness we may add the two extreme cases of the completely free and the completely fixed point. In the former the three supplementary equations are the equations $\mathbf{Z} = 0$; in the latter the equations $\mathbf{q} = 0$. From (216) we then get in the first case the motion of the point, in the second case the magnitude and direction of the constraining force $\mathbf{Z}$ which keeps it fixed.

The greater the number of prescribed conditions, the less is the number of independent variables, the so-called free co-ordinates. Hence we find it appropriate to speak of a greater or lesser freedom of motion of the point and we set the number of free co-ordinates equal to the number of degrees of its freedom of motion. A point has 3, 2, 1, 0 degrees of freedom of motion according as it is free, or is movable on a surface or a curve, or is fixed.

The above discussion completes the essential features of the theory of motion of a point which is not free. We have now only to consider the most important applications.

§ 66. The general equations of motion (216) are often written in the more symmetrical form :

$$\boldsymbol{F} + \boldsymbol{Z} - m\dot{\boldsymbol{q}} = 0 \quad . \quad . \quad . \quad . \quad (217)$$

which is enunciated as the theorem : if we imagine that besides the propelling force $\boldsymbol{F}$ and the constraining force $\boldsymbol{Z}$ there is a third force $-m\dot{\boldsymbol{q}}$, which also acts on the reference-point, then these three forces maintain equilibrium among themselves. However slight this change of expression may seem it is nevertheless very important in applications owing to its convenience and vividness; it has therefore received a special name as the " Principle of d'Alembert." * It refers the laws of motion quite generally to the laws of equilibrium, but of course it adds nothing substantially to the Newtonian equations. The fictitious force $-m\dot{\boldsymbol{q}}$ is usually called the " inertial resistance."

Instead of resolving the forces along the fixed co-ordinate-directions x, y, z, we may also resolve them, as in § 25, along the directions of the tangent τ, the principal normal ν and the binormal β of the orbital curve of the reference-point, and so obtain the following expression for d'Alembert's Principle, if we take into account (74a) and (75) as well as the fact that the constraining force has no component in the direction of the tangent :

$$\boldsymbol{F}_\tau - m\frac{dq}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

$$\boldsymbol{F}_\nu + \boldsymbol{Z}_\nu - \frac{mq^2}{\rho} = 0 \quad . \quad . \quad . \quad . \quad . \quad (219)$$

$$\boldsymbol{F}_\beta + \boldsymbol{Z}_\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

Here τ must be taken in the direction of the velocity, ν in the direction pointing towards the centre of curvature.

A question which has often been raised and discussed vigorously is whether the inertial resistance or its component, the centrifugal force, is a " real " force. It is easy

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* The name d'Alembert's Principle is often also given to the equation (383) which results from the combination of (217) with the Principle of Virtual Work (321).

to answer this question as soon as we have come to a decision about the definition of force, which is arbitrary from the very beginning. If, as in § 9, we set the force in the same direction as and proportional to the acceleration, then the inertial resistance is *not* a real force: for the inertial resistance is *not* in the same direction as and proportional to the acceleration. But if we modify the definition of force (and no objection can be raised against this) in such a way that all the forces are always in equilibrium, then the inertial resistance must also be reckoned as a force. The essential feature is not the names but the equations (217) and they certainly do not contain the slightest indefiniteness.

§ 67. We shall call attention at once here to an important property of a constraining force. Since the component of the constraining force is zero in the direction of the velocity so also the *work performed by the constraining force* (§ 47) *is equal to zero*, and, clearly, this theorem also holds for the two extreme cases of the free and the fixed point, because in the first case $\boldsymbol{Z} = 0$ and in the second case $\boldsymbol{q} = 0$.

A number of important consequences follow from this. Let us first imagine the reference-point to be at rest, situated on a fixed surface or curve and acted upon by a given propelling force $\boldsymbol{F}$.

In general it will begin to move—namely, in the direction of the resultant $\boldsymbol{F} + \boldsymbol{Z}$. Hence the work performed by the total force in the initial displacement, whose amount is $d\boldsymbol{r}$, is positive:

$$(\boldsymbol{F} + \boldsymbol{Z}) \,.\, d\boldsymbol{r} > 0.$$

But since $\boldsymbol{Z} \,.\, d\boldsymbol{r} = 0$, it follows that:

$$\boldsymbol{F} \,.\, d\boldsymbol{r} > 0 \quad . \quad . \quad . \quad . \quad . \qquad (221)$$

That is, if a free or a constrained reference-point at rest is set into motion by a propelling force, the work of the propelling force is positive, or the initial displacement forms an acute angle with the propelling force.

If the propelling force has a potential U, then, in view of (150), we have :

$$dU < 0 \quad . \quad . \quad . \quad . \quad . \quad (222)$$

that is, the potential decreases when the motion begins. From this a sufficient condition for equilibrium immediately follows. For if among all the directions of displacement of which the movable reference-point is susceptible in consequence of the prescribed conditions there is not one for which the potential decreases, no motion can occur and the reference-point must remain at rest.

This is realized when, for example, the reference-point is situated at a point on its surface or curve where the potential U is a maximum or a minimum. For in that case $\delta U = 0$ for every displacement, and so the inequality (222) cannot be satisfied. Hence the reference-point is then in equilibrium. But we also see further that if U is a maximum the equilibrium is unstable. For if the reference-point is displaced slightly from its position of equilibrium and is then again released, it will, of course, begin to move. But since according to (222) the motion takes place in the sense of decreasing potential, it is impossible for the reference-point to return to its position of equilibrium, which corresponds to the maximum. Conversely, the equilibrium is stable at a position corresponding to a maximum of the potential. But if the potential remains constant over a finite range of displacements, the equilibrium is neutral (*indifferent*). For then the reference-point is in equilibrium at every point because there is no possibility by which it may be enabled to fulfil the condition (222) which is necessary if motion is to occur.

A vivid example of these theorems is given by a heavy point on a fixed surface or curve. Here U is given by (152), and the condition (222) becomes :

$$dz < 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

Hence the motion always starts in the downward direction. If the height of the surface or the curve has a maximum or a minimum anywhere, the reference-point is in stable or unstable equilibrium at those points; if the surface or curve runs horizontally for some distance, the reference-point is in neutral equilibrium along that distance.

If we now consider, instead of a reference-point which was originally at rest, one which is moving with arbitrary velocity, the theorem that the work of the constraining force vanishes also holds in this case, and hence of the work of the whole force only that of the propelling force remains.

Hence the equation (147) *of vis viva holds equally well for it, exactly as if the constraining force and the prescribed conditions were not present at all;* and if the propelling force has a potential the integral principle (151) of vis viva also holds.

A heavy point-mass which can move on a fixed surface or curve thus always has a definite velocity at a definite height, no matter when, where and by which route it has arrived at this height. The greater the height the smaller the velocity.

All the theorems developed in this section are capable of being considerably generalized; these generalizations will be discussed in the second chapter of the second part of the present volume, and serve as a good basis for understanding those extensions.

We proceed to discuss special cases, beginning with the simplest: a single straight line as the only degree of freedom.

§ 68. **Fixed Curve.** In addition to the equations of motion (217) we have also the two equations to the curve and the condition that the constraining force is perpendicular to the curve:

$$\mathbf{Z}_x \frac{dx}{ds} + \mathbf{Z}_y \frac{dy}{ds} + \mathbf{Z}_z \frac{dz}{ds} = 0 \quad . \quad . \quad . \quad (224)$$

where the direction-cosines of the curve-element ds are to be regarded as given.

Let us next inquire into the condition that the reference-point be in equilibrium under the influence of a given propelling force F. Then the acceleration is zero and by eliminating Z from (217) and (224) we get as the condition for equilibrium :

$$F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} = 0 \quad . \quad . \quad . \quad (225)$$

Thus the propelling force need not vanish, as in the case of a free point, but it is sufficient if it acts at right angles to the curve.

We next inquire into the motion of the reference-point in the case where the driving force is equal to zero. Then it follows from (218) that :

$$q = \text{const.},$$

that is, the velocity is constant. From (219) and (220) :

$$Z_\beta = 0, \; Z_\nu = \frac{mq^2}{\rho},$$

that is, the constraining force coincides with the centripetal force. For a fixed straight line the constraining force is equal to zero.

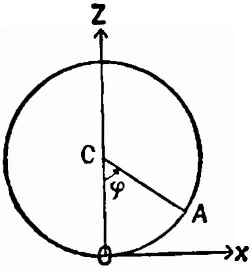

FIG. 15.

§ 69. We now consider the motion of a heavy point along a fixed vertical circular arc—that is, a *circular pendulum*. This is realized most simply by a rigid weightless rod, which can rotate about a fixed point in a vertical plane and which carries the point-mass at its free end.

As usual, we take the vertical as the z-axis, the plane of the circle as the xz-plane, and the origin of co-ordinates O at the lowest point of the circle, whose radius, the length of the pendulum, we assume to be l (Fig. 15).

Then the equations to the circle are :

$$y = 0 \quad \text{and}$$

$$x^2 + z^2 = 2lz \quad . \quad . \quad . \quad . \quad . \quad (226)$$

Since the reference-point has only one independent co-ordinate, it is sufficient for calculating the velocity to integrate the equations of motion once. We find it convenient, by § 67, to use the principle of vis viva (152*a*):

$$q^2 + 2gz = q_0^2 \quad . \quad . \quad . \quad . \quad . \quad (227)$$

where q_0 denotes the velocity for $z = 0$.

Since there is a definite velocity q corresponding to every height h, the motion is periodic. But its character is quite different according to whether the velocity q_0 with which the reference-point leaves its stable position of equilibrium is sufficient to bring it into the unstable state of equilibrium, $z = 2l$, or not. In the former case the oscillations of the pendulum are all in the same sense; in the latter case the pendulum comes to rest at a height $z < 2l$ and the oscillations occur alternately to and fro. The limiting case, however, is that where the initial velocity q_0 just suffices to reach the greatest height $z = 2l$ with the velocity $q = 0$, and so, by (227):

$$q_0 = 2\sqrt{lg} \quad . \quad . \quad . \quad . \quad . \quad (228)$$

In that case the reference-point remains at rest in the uppermost position. But if:

$$q_0 < 2\sqrt{lg} \quad . \quad . \quad . \quad . \quad . \quad (229)$$

then the pendulum comes to rest for:

$$z = \frac{q_0^2}{2g} \; (< 2l) \quad . \quad . \quad . \quad . \quad (230)$$

and then reverses.

The constraining force $\boldsymbol{Z}$ in this motion is represented by the pull inwards or the pressure outwards, which the rod exerts on the reference-point A, being positive when it acts towards the centre of curvature C—that is, inwards. Actually, in (220), the direction β, the binormal coincides with y, and since $\boldsymbol{F}_y = 0$, $\boldsymbol{Z}_y$ also vanishes, and the whole constraining force $\boldsymbol{Z}$ acts in the direction of the radius.

Since, further, $\rho = l$ and $\boldsymbol{F}_\nu = -mg\frac{l-z}{l}$, the component of the weight in the direction towards C, it follows from (219) that:

$$\boldsymbol{Z}_\nu = \frac{mq^2}{l} + mg\frac{l-z}{l}.$$

A positive value for $\boldsymbol{Z}_\nu$ may also be realized by taking an inextensible thread instead of the rigid rod. But if $\boldsymbol{Z}_\nu$ comes out negative, a thread no longer suffices to maintain the condition for rigidity, and we are then compelled to take the incompressible rod. The last equation shows that for $z < l$—that is, in the lower half of the circle—$\boldsymbol{Z}_\nu$ is always positive. In this case, therefore, a thread or string is sufficient under all circumstances. In general, we obtain by eliminating q from (227):

$$\boldsymbol{Z}_\nu = \frac{m}{l}(q_0^2 + g\,.\,[l - 3z]) \quad . \quad . \quad . \quad (231)$$

As z increases $\boldsymbol{Z}_\nu$ decreases.

Let us take again the limiting case (228) considered above, in which the reference-point just reaches its highest point. Then:

$$\boldsymbol{Z}_\nu = \frac{mg}{l}(5l - 3z) \quad . \quad . \quad . \quad . \quad (232)$$

that is, the constraining force remains positive up to the height $z = \frac{5l}{3}$; after that the pull becomes a pressure, and if the reference-point attains the height $2l$ with the velocity zero, the pressure has become equal to $-mg$, corresponding to the weight of the point, which is now at rest.

For very great values of the initial velocity q_0 we see, by (231), that $\boldsymbol{Z}_\nu$ always remains positive; hence a thread is then sufficient to swing the pendulum around, as in the case of a sling. The smallest permissible value of q_0 for this is obtained from the condition that $\boldsymbol{Z}_\nu$ has its lowest value, zero, at the highest point, $z = 2l$:

$$q_0^2 = 5lg \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

which is of course a little greater than the limiting value (228).

But a thread will suffice instead of the rod even for the vibratory motion provided the initial velocity q_0 is sufficiently small—namely, when q_0 is smaller than that initial velocity for which $\boldsymbol{Z}_v$ is zero in the highest position (230). By (231) this limiting value is :

$$q_0^2 = 2lg \tag{234}$$

Hence it is only when q_0^2 lies between the limits (233) and (234) that a rod is necessary to keep the reference-point in a circular orbit. In other cases an inextensible thread is sufficient.

§ 70. Let us next inquire into the relationship between space and time. For this purpose a second integration is necessary, for which we find it expedient to introduce the angle of displacement ϕ (Fig. 15) by means of the equations :

$$q = \pm l, \frac{d\phi}{dt} \qquad \frac{l - z}{l} = \cos\phi \quad . \tag{235}$$

Then we easily obtain from (227) :

$$t = \int_0^\phi \frac{l \, . \, d\phi}{\sqrt{q_0^2 - 4lg \sin^2 \dfrac{\phi}{2}}} \quad . \quad . \quad . \tag{236}$$

where we have set $t = 0$ for $\phi = 0$.

This elliptic integral reduces to an elementary function only for the limiting case (228), in which the initial velocity q_0 is just sufficient to bring the pendulum into the unstable position of equilibrium ($\phi = \pi$). The time which passes until then comes out from (236) as having an infinite logarithmic value ; this arises from the circumstance that the velocity finally becomes vanishingly small.

We now further investigate the more important case of oscillations to and fro (vibrations)—that is, we assume the inequality (229) to be fulfilled. By (230) the pendulum then comes to rest for the angle of displacement ϕ_1, when by (230) and (235):

$$\sin^2\frac{\phi_1}{2}=\frac{q_0^2}{4lg} \quad . \quad . \quad . \quad . \quad . \quad (237)$$

ϕ_1 is the amplitude of the vibration. If we introduce ϕ_1 in place of q_0 in (236) we get:

$$t=\frac{1}{2}\sqrt{\frac{l}{g}}\int_0^\phi\frac{d\phi}{\sqrt{\sin^2\frac{\phi_1}{2}-\sin^2\frac{\phi}{2}}} \quad . \quad . \quad (238)$$

Since ϕ increases and decreases periodically as the time increases, the square root is to be taken alternatively positive and negative. From now onwards we restrict our attention to the first upward swing of the pendulum—that is, the first quarter of the first vibration. Then the square root is positive, and likewise ϕ.

To bring the integral into its normal form we introduce instead of ϕ the integration variable θ by means of the relationship:

$$\sin\frac{\phi}{2}=\kappa\,.\sin\theta \quad . \quad . \quad . \quad . \quad (239)$$

where we use the abbreviation:

$$\sin\frac{\phi_1}{2}=\kappa \quad . \quad . \quad . \quad (240)$$

We then get from (238):

$$t=\sqrt{\frac{l}{g}}\,.\int_0^\theta\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}} \quad . \quad . \quad . \quad (241)$$

The reciprocal of the square root may be expanded as a series which converges the more rapidly the smaller the amplitude; the integration may then be performed term by term. If we stop the series at the term containing κ^2, the approximate value:

$$t=\sqrt{\frac{l}{g}}\left\{\theta+\frac{\kappa^2}{4}\left(\theta-\frac{\sin 2\theta}{2}\right)\right\} \quad . \quad . \quad (242)$$

follows.

The first quarter-vibration ends when $\phi = \phi_1$, and consequently when θ has become equal to $\frac{\pi}{2}$. Hence if T denotes the time of a whole vibration we have :

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\kappa^2}{4}\right),$$

and by (240), since the sine may be replaced by its arc as a first approximation, we have :

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\phi_1{}^2}{16}\right) \quad . \quad . \quad (243)$$

For infinitely small amplitudes the time of vibration is completely independent of the amplitude; but even for amplitudes of only a few degrees the term in ϕ_1 is very small.

If we wish to restrict ourselves from the very beginning to infinitesimal amplitudes, it is better, in deriving the laws of vibration, to start directly from the equation (218), which for its application to the present case runs, by (235):

$$g \sin \phi + l\frac{d^2\phi}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad (244)$$

If we replace sin ϕ by ϕ here we have precisely the differential equation (15), and we may take over directly the results there obtained.

The laws which we have found for the infinitely small vibrations of the circular pendulum may easily be generalized to apply to the vibrations of a heavy point along any arbitrary curve that lies in a vertical plane about its position of stable equilibrium. For since in these vibrations the reference-point moves to only an infinitesimal extent from its position of equilibrium, only the infinitely adjacent points of the curve come into question in the process, and hence the laws for the circular pendulum apply here too, except that, in place of the circular radius l, we now take the radius of curvature of the curve at its lowest point. In the case of finite vibrations, however, the further course of the curve has an influence. If the radius of curvature l is constant throughout, then by (243) the time of vibration increases as the amplitude increases. But if the curvature of the curve increases with the height —that is, if the curve ascends more rapidly than the circle of curvature at its lowest point, the time of vibration becomes less than in the case of the circle, and by choosing the curvature appropriately we may also succeed in making the time of vibration independent of the amplitude even for finite vibrations. (This curve, called the "tautochrone," is the ordinary cycloid which is produced by rolling a circle of radius $\frac{l}{4}$ on a straight line; any point on the circle then traces out a cycloid.)

§ 71. **Fixed Surface.** For a point-mass which is constrained to remain on a given fixed surface we have, by § 65, besides the equation of motion (217) also the equation to the surface:

$$f(x, y, z) = 0 \quad . \quad . \quad . \quad . \quad . \quad (245)$$

and secondly, the condition that the constraining force acts perpendicularly to the surface, that is, in the direction of its normal:

$$\boldsymbol{Z}_x : \boldsymbol{Z}_y : \boldsymbol{Z}_z = \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} \quad . \quad . \quad . \quad (246)$$

From these equations all the laws of the motion, and also the magnitude and direction of the constraining force, may be derived uniquely.

Let us first again inquire into the condition that the reference-point be in equilibrium under the influence of a given driving force. Then the acceleration is zero, and by eliminating $\boldsymbol{Z}$ from (217) and (246) we get as our condition of equilibrium:

$$\boldsymbol{F}_x : \boldsymbol{F}_y : \boldsymbol{F}_z = \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} \quad . \quad . \quad . \quad . \quad (247)$$

This represents two equations, whereas for the equilibrium of a point on a fixed curve only the single equation of condition (225) must be fulfilled.

If we add that in the extreme case of a free point the three equations $\boldsymbol{F} = 0$ and that in the opposite extreme case of a fixed point no condition at all is necessary for equilibrium, then it is clear that in each of all the cases quoted the number of equations of condition for equilibrium agrees exactly with the number of degrees of freedom of the point (§ 65)—a theorem which will be considerably generalized later.

For the motion of a point on a fixed surface in the case where the driving force $\boldsymbol{F} = 0$, we get from (218):

$$\frac{dq}{dt} = 0, \quad q = \text{const.} \qquad (248)$$

and from (219) and (220):

$$|\boldsymbol{Z}| = \frac{mq^2}{\rho}. \qquad (249)$$

exactly as in the case of motion under no forces (*kräftefrei*) on a fixed curve. There is an essential difference, however, in that here neither the radius of curvature ρ nor the orbital curve is at all known from the outset, but that it must first be found. For the initial state gives us only the position and the tangent to the orbit: the further course of the curve on the surface $f = 0$ is to be specially calculated.

For this purpose we have the equations (246), which, in view of the fact that the constraining force is the only force which is acting here, give, by (68) and (248):

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} = \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} \qquad (250)$$

that is, the principal normal at any point of the orbital curve coincides with the normal to the surface at this point. This is a particular property of the orbital curve, which does not belong to every curve drawn on the surface. For example, on a spherical surface the principal normal of any arbitrary circular section is the radius of this circle, whereas the surface normal is the radius of the sphere. A curve with the distinguishing property (250) is called a "geodetic line" of the surface; the name arises from a further important property of these curves which is to be derived later (§ 111). Hence according to what has been said, the geodetic lines are the great circles, for the plane they are the straight lines; for the normal to the plane is the direction perpendicular to the plane, whereas for every plane curve which is not a straight line, the principal normal lies in the plane. Hence a point-mass which is not subject to a driving force moves on a fixed surface along a geodetic line with constant velocity. The orbit is determined by the initial state; for there is only one geodetic line on the surface which passes through a definite point and has a definite tangent. This is seen most clearly if we reflect that the first element of the curve and the known normal to the surface at the end-point of the element determine the plane of curvature of the curve, and the point of intersection of this plane with the surface determines the second element of the curve, and so forth, successively.

So a point-mass under no forces moves on a sphere in the great circle and on a plane in that straight line which is determined by the direction of the initial velocity. The magnitude of the force of constraint is in each case given by (249).

§ 72. We next consider the motion of a heavy point on a fixed spherical surface—that is, a *spherical pendulum*. This is realized most simply by a rigid weightless rod which can be turned about a fixed point in all directions and which carries the point-mass at its free end. By (246) the direction of the constraining force coincides with the radius of the sphere. If it acts in the direction of the centre of the sphere, the rigid rod may be replaced by an inextensible thread.

In the following remarks we restrict ourselves to determining only the motion of the pendulum. Let us again take the origin at the lowest point of the sphere and the z-axis vertically upwards; then the equation to the sphere of radius l is:

$$x^2 + y^2 + (l - z)^2 = l^2 \quad . \quad . \quad . \quad (251)$$

In addition we have, by § 67 and (152a) the equation of vis viva:

$$q^2 + 2gz = c \quad . \quad . \quad . \quad . \quad . \quad (252)$$

Besides these we here also need a second integral of the equations of motion; for this we may by § 51 use the principle of sectorial areas in its extended form. For the total force acting on the reference-point—that is, the resultant of the weight and the force of constraint—does not, indeed, go through a fixed centre, but through a fixed straight line—namely, through the vertical at the centre of the sphere. Hence the equation (161) holds for the projection of the reference-point on the xy-plane:

$$r^2 \frac{d\phi}{dt} = c' \quad . \quad (253)$$

where:

$$x = r \cos \phi, \; y = r \sin \phi \quad . \quad . \quad . \quad (254)$$

To determine the orbit we introduce everywhere instead of the rectilinear co-ordinates x, y, z the cylindrical co-ordinates r, ϕ, z. Then (251) becomes:

$$r^2 + z^2 = 2lz \quad . \quad . \quad . \quad . \quad . \quad (255)$$

and (252), in view of (166), becomes:

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + \left(\frac{d}{dt}\right)^2 + 2gz = c \quad . \quad . \quad (256)$$

Further, if we eliminate dt by means of (253), and r^2 and rdr by means of (255), and use the differential equation derived from (255):

$$rdr = (l - z)dz \quad . \quad . \quad . \quad . \quad (257)$$

we get the following relation between ϕ and z:

$$d\phi = \frac{l \,.\, c' \,.\, dz}{(2l - z) \,.\, z \,.\, \sqrt{(c - 2gz) \,.\, (2l - z) \,.\, z - c'^2}} \quad . \quad (258)$$

In general, this leads to an elliptic integral. Since by (253) ϕ always changes in the same sense as t, we may, without essential loss of generality, assume ϕ to be always increasing and hence assume c' and likewise c as positive. On the other hand, z will alternately decrease and increase, and the square root in (258) will have to be taken positive or negative correspondingly. The vanishing of the root gives the highest and the lowest positions of the pendulum. The equation for this is cubic in z and so has three roots, but it is easy to show that one root is greater than $2l$ and hence is of no significance physically. For the expression under the root changes sign if z is made to increase from $2l$ to ∞.

Of course, z is periodic with respect to ϕ. But the orbital curve is closed only if a whole multiple of the period of ϕ is equal to a whole multiple of 2π—that is, if the ratio of this period to π is rational.

If the maximum and the minimum of z coincide or if the two roots of the cubic equation that come into question are equal then the pendulum remains constantly at the same height z and executes horizontal circular vibrations; in that case r and $\frac{d\phi}{dt}$ are also constant. We obtain these values by differentiating the expression under the root in (258) with respect to z and setting the result equal to zero:

$$(c - 2gz)(l - z) - g(2lz - z^2) = 0$$

or, by (252) and (255):

$$q^2(l - z) - gr^2 = 0 \quad . \quad . \quad . \quad . \quad (259)$$

This equation can be satisfied for any arbitrary value of z between 0 and l—that is, on the lower half of the sphere. The corresponding value of r is obtained from (255), and then q and the angular velocity from (259):

$$\frac{d\phi}{dt} = \frac{q}{r} = \sqrt{\frac{g}{l - z}} \quad . \quad . \quad . \quad . \quad (260)$$

For $z = l$ the angular velocity becomes infinite; for infinitely small values of z it assumes a definite finite value —namely, precisely that which corresponds to the period of vibration (243) of a circular pendulum of infinitesimal amplitude.

The constraining force—that is, the tension of the pendulum thread—is, by § 66, in equilibrium with the centrifugal force $\frac{mq^2}{r}$ and the gravitational force mg. The resultant of the last two forces passes through the centre of the sphere, since their quotient, by (259), is equal to the tangent of the angle of displacement: $\frac{r}{l-z}$ (Fig. 16).

The magnitude of the tension is:

$$m\sqrt{\frac{q^4}{r^2}+g^2}=\frac{mlg}{l-z} \quad . \quad . \quad (261)$$

For $z=0$, it becomes equal to mg, for $z=l$ it becomes infinite, as is evident.

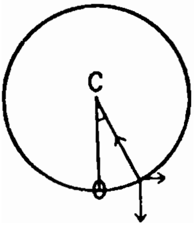

FIG. 16.

§ 73. If we now consider infinitesimal vibrations of a spherical pendulum in general, we may again start out from (258) and introduce into it the simplifications which are characteristic of infinitesimal vibrations. For this purpose we assume that both r, the displacement, and q, the velocity, are infinitesimals of the first order, whereas ϕ and $\frac{d\phi}{dt}$ may be finite. By (255) the height z, and by (252) and (253) the constants c and c', are infinitesimals of the second order, and hence it follows that for the whole duration of the motion all these orders of magnitude persist.

Hence the curve of the reference-point coincides as far as quantities of the second order with its projection on the xy-plane—that is, it is approximately a horizontal plane curve, and the equation (255) becomes simplified to:

$$r^2=2lz \quad . \quad . \quad . \quad . \quad . \quad (262)$$

If we now consider the differential equation (258) in the light of the above restrictive simplifications we find that the factor $2l - z$ may be replaced to a high degree of approximation by $2l$. But this is the only permissible simplification; for otherwise the individual terms in all the sums and differences have the same order of magnitude. Hence we now obtain as the differential equation for infinitesimal vibrations:

$$d\phi = \frac{c'dz}{2z\sqrt{2lz(c - 2gz) - c'^2}}.$$

Integrated this gives:

$$\phi = \frac{1}{2}\cos^{-1}\frac{lcz - c'^2}{z\sqrt{l^2c^2 - 4lgc'^2}} \quad . \quad . \quad . \quad (263)$$

where we set the integration constant equal to zero for the same reason as in integrating (168).

Since the curve lies very approximately in the xy-plane, we introduce in (263) in place of ϕ and z the rectangular co-ordinates x and y, by (262) and (254), and so obtain the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad (264)$$

where a^2 and b^2 are the two values which the expression:

$$\frac{2lc'^2}{lc \pm \sqrt{l^2c^2 - 4lgc'^2}} \quad . \quad . \quad . \quad (265)$$

has, according as the root is given the one or the other sign. Thus the curve is an ellipse with semi-axes a and b. To discover how this ellipse is traversed, we set:

$$x = a\cos\theta, \; y = b\sin\theta \quad . \quad . \quad . \quad (266)$$

which satisfy equation (264) identically, and find how the angle θ depends on the time t. We get from (253):

$$r^2\frac{d\phi}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = ab\frac{d\theta}{dt} = c'$$

Hence:

$$\theta = \frac{c'}{ab}\cdot t \quad . \quad . \quad . \quad . \quad . \quad . \quad (267)$$

if we set $\theta = 0$ for $t = 0$: this is a very simple relationship. If the values of a and b are substituted from (265), we get:

$$\theta = \sqrt{\frac{g}{l}} \cdot t,$$

and this value, inserted in (266), gives:

$$x = a \cos t\sqrt{\frac{g}{l}} \text{ and } y = b \sin t\sqrt{\frac{g}{l}} \cdot \qquad (268)$$

which determines the motion in all its details. Then a and b are given by (265) and z by (262). The period of vibration is independent of a and b and is the same as in the case of the circular pendulum of infinitesimal amplitude.

It is interesting to observe that the motion here found agrees exactly with that of a freely moving point-mass which is in the xy-plane and is acted on by a certain central force emanating from O. For such a motion, as we saw in § 52, is determined by the two equations of the principle of sectorial areas and the principle of vis viva. The former is fulfilled by (253), but the latter is also to be regarded as fulfilled if we write the equation (252), in view of (262), in the form:

$$\frac{mq^2}{2} + \frac{mgr^2}{2l} = \text{const.} \quad . \quad . \quad . \quad . \qquad (269)$$

and if we recollect, on the other hand, that for the central motion which we have assumed the principle of vis viva holds in the following form, which follows from (151), (109) and (108):

$$\frac{1}{2}mq^2 + \int f(r)dr = \text{const.},$$

where $f(r)$ denotes the attractive force in magnitude and sign. Comparison with (269) gives:

$$f(r) = \frac{mg}{l} \cdot r \quad . \quad . \quad . \quad . \quad . \quad . \qquad (270)$$

that is, the force is an attractive one and is proportional to the distance from O. Of course, we may also show directly that this law of attraction leads to the equations of motion (268), not only for infinitesimal vibrations, but also for arbitrarily great vibrations.

§ 74. We shall now investigate the influence of the earth's rotation on the vibrations of a spherical pendulum. For this purpose the equations (215) are available; they express the laws of motion of a material point m for an observer situated on the earth's surface at the geographical latitude β, the point m being acted on by a force, whose components are X, Y, Z, in addition to its own weight. If S is the tension of the thread, then:

$$X = -S\cdot\frac{x}{l},\quad Y = -S\cdot\frac{y}{l},\quad Z = -S\cdot\frac{z-l}{l},$$

and the equations of motion run:

$$\left.\begin{array}{ll}\text{Southwards:} & m\dot{u} = -S\cdot\frac{x}{l} + 2m\omega v\sin\beta \\ \text{Eastwards:} & m\dot{v} = -S\cdot\frac{y}{l} - 2m\omega(u\sin\beta + w\cos\beta \\ \text{Towards the zenith:} & m\dot{w} = -S\cdot\frac{z-l}{l} - mg + 2m\omega v\cos\beta\end{array}\right\}\quad(271)$$

These equations together with (251) contain the complete solution to the problem.

We now inquire whether the principles of vis viva and of sectorial areas still hold here. For this purpose we multiply the equations of motion successively, as in § 47, by u, v, w, then add and integrate. We find that the terms in S, the terms in ω and also those in r cancel out; the former because the equation (251) holds for all times, and hence may also be differentiated with respect to t; and so the principle of vis viva is found to be valid in precisely the form (252).

There still remains the principle of sectorial areas. As in § 50 we multiply the first equation of motion by y, the second by x, and subtract. Then:

$$x\dot{v} - y\dot{u} = -2\omega(xu\sin\beta + xw\cos\beta + yv\sin\beta)$$

We now again assume the vibrations to be infinitely small. Then w becomes an infinitesimal of the second order compared with u and v. So if we omit the term in w we get, on integration:

$$r^2 \frac{d\phi}{dt} = -\omega r^2 \sin\beta + \text{const.} \quad . \quad (272)$$

Hence the principle of sectorial areas does not hold here. We can, however, obtain a picture of the motion by making a simple substitution. For if we set:

$$\phi' = \phi + \omega \sin\beta \, . \, t \quad . \quad . \quad . \quad (273)$$

then (272) becomes:

$$r^2 \frac{d\phi'}{dt} = \text{const.} \quad . \quad . \quad . \quad . \quad (274)$$

That is, the principle of sectorial areas holds for a co-ordinate system which rotates about the z-axis (the vertical) with the angular velocity $-\omega \sin\beta$. For when ϕ' is constant we have:

$$\frac{d\phi}{dt} = -\omega \sin\beta.$$

Hence if we refer the vibrations of the pendulum to this rotating co-ordinate system we find that exactly the same laws hold as were deduced in § 73 for a system absolutely at rest. To prove this we now require to prove only that the principle of vis viva is valid also for the rotating system. For this principle, together with the principle of sectorial areas, determines uniquely the motion on the spherical surface. If we write (269) in polar co-ordinates and neglect w^2 we have:

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + \frac{gr^2}{l} = \text{const.} \quad . \quad . \quad (275)$$

and if, by (273), we introduce ϕ' in place of ϕ:

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi'}{dt}\right)^2 - 2\omega \sin\beta r^2 \frac{d\phi'}{dt} + r^2\left(\omega^2 \sin^2\beta + \frac{g}{l}\right) = \text{const.},$$

a relation which, if (274) is taken into consideration, has exactly the form (275), except that now ϕ' takes the place of ϕ and the constants are slightly altered.

Hence we may enunciate the theorem that, relatively to the rotating earth, the vibrations of the pendulum of infinitely small amplitude occur in the same way as relatively to the fixed earth—that is, in an ellipse, except that the axes of the ellipse rotate with the angular velocity $-\omega \sin \beta$—that is, in the direction south-west-north-east on the northern hemisphere ($\beta > 0$) and in the reverse direction on the southern hemisphere. The phenomenon vanishes entirely at the equator and its effect is a maximum at the poles.

Foucault's famous pendulum experiment confirms the theory.

§ 75. We shall now investigate the case where the prescribed conditions are *dependent on the time*—that is, where the reference-point is constrained to remain on a curve or a surface, which moves in some given way.

The equations $f = 0$ and $\phi = 0$ then contain, besides the co-ordinates x, y, z of the reference-point, also the time explicitly. The problem of determining the motion of the reference-point for a given driving force $\boldsymbol{F}$ may be treated on exactly the same lines as those described in § 65. Exactly the same result is obtained—that is, the motion is determined by the three equations of motion (216) or by d'Alembert's principle (217) in conjunction with the additional three equations which express the prescribed conditions and the theorem that the constraining force $\boldsymbol{Z}$ is directed at right angles to the curve or surface.

On the other hand, the theorems derived in the later §§ 66 and 67 lose their validity here in general. In particular, it is no longer correct to assume that the constraining force $\boldsymbol{Z}$ is directed perpendicularly to the tangent of the reference-point. For, in general, the orbital curve has a different tangent from that to the prescribed curve or surface at the same point.

A simple example will make this clear. Suppose the reference-point is constrained to remain on a straight line which rotates with a given angular velocity in a horizontal plane. Let Ot be the position of the straight line at the time t, Ot' its position at the infinitely near time t' (Fig 17). Let the reference-point be situated at A at the time t and at A' at the time t'. Then the tangent of the orbital curve is AA', whereas the tangent of the prescribed curve is AB, and in general these two directions make a finite angle with each other. Since the constraining force $\mathbf{Z}$ acts perpendicularly to AB, it will in general form an acute or an obtuse angle with AA'. Hence it follows that the work done by the constraining force is not, as in § 67, equal to zero, and also that the principle of vis viva is not in general obeyed, even when the driving force has a potential.

Let us perform the calculation for the simple example we have chosen, under the assumption that the angular velocity ω is constant and that no driving force is acting. We take the central point O of the rotation as the origin of co-ordinates and the plane of rotation as the (x, y)-plane. Then by (216) the equations of motion are:

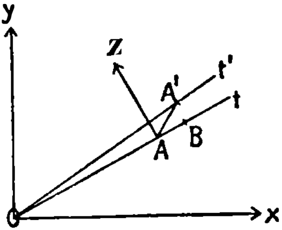

FIG. 17.

$$m\dot{u} = \mathbf{Z}_x,\ m\dot{v} = \mathbf{Z}_y \quad . \quad . \quad . \quad . \quad (276)$$

The equation expressing the prescribed condition is:

$$y = x \tan(\omega t) \quad . \quad . \quad . \quad . \quad . \quad (277)$$

and the theorem concerning the direction of the constraining force is:

$$x\mathbf{Z}_x + y\mathbf{Z}_y = 0 \quad . \quad . \quad . \quad . \quad . \quad (278)$$

These expressions and the initial state determine the motion. We get by eliminating $\mathbf{Z}_x$ and $\mathbf{Z}_y$:

$$x \,.\, \dot{u} + y \,.\, \dot{v} = 0 \quad . \quad . \quad . \quad (278a)$$

and, by introducing polar co-ordinates r and ϕ, according to (254), and taking into account (277):

$$\ddot{r} - \omega^2 r = 0 \quad . \quad . \quad . \quad . \quad . \quad (278b)$$

This equation may be integrated term for term if we multiply by $\dot{r}$: this leads to:

$$\frac{1}{2}\dot{r}^2 - \frac{1}{2}\omega^2 r^2 = \text{const.}$$

Let us assume that in the initial state $r = a$ and $\dot{r} = 0$; then we get the value of the constant of integration and hence the differential equation:

$$dt = \frac{dr}{\omega\sqrt{r^2 - a^2}}.$$

Integrating this we have:

$$r = \frac{a}{2}(e^{\omega t} + e^{-\omega t}) \quad . \quad . \quad . \quad . \quad (279)$$

Thus the reference-point is flung outwards with ever-increasing velocity—which is easy to understand from the circumstance that the constraining force always performs positive work, as we see from Fig. 17. From (279) and (277) we obtain as the orbital curve:

$$r = \frac{a}{2}(e^{\varphi} + e^{-\varphi}) \quad . \quad . \quad . \quad . \quad (280)$$

a logarithmic spiral, whose shape is independent of ω.

It suggests itself to ask what the position is with regard to the universally valid principle of conservation of energy (§ 49), since the mechanical principle of vis viva is here transgressed. The energy principle, of course, also remains valid here; the vis viva of the reference-point by no means arises from nothing, but is supplied by the work of the source of power which occasions the rotation of the straight line. For in order to maintain the prescribed condition that the angular velocity remain constant power is necessary which must be supplied from outside, and this will be the greater the further the point-mass moves outwards. The work performed by this force is, by the principle of energy, exactly equal to the increase of the vis viva of the material reference-point.

More generally we may say that in every case where a prescribed condition contains the time explicitly, a certain amount of external work will have to be performed to maintain this condition, whereas the conditions which are independent of the time need no performance of external work to be realized; this corresponds with the circumstance that the work of the constraining force is zero in their case (§ 67).

PART TWO

MECHANICS OF A SYSTEM OF MATERIAL POINTS

CHAPTER I

STATICS OF A RIGID BODY

§ 76. In nature we have to deal, not with material points, but with material bodies of finite extent. But we may regard every point as composed of very many material points and we may trace the differences in the mechanical properties of bodies back to the circumstance that their individual points act on one another with different forces. This reduces the question of the equations of motion of material bodies to that of the mechanics of systems of material points.

From this point of view there are in nature no other mechanical forces at all except those between material points. Every material point moves in response to the resultant of the forces which are exerted on it by all the remaining points of the universe. Whenever we speak of a force which a whole body exerts or experiences, we do not mean this to be taken literally, but only as an abbreviated mode of expression. In reality only the individual points are the places where the forces originate or act. For every force acts from one definite material point A to a second definite material point B.

Hence all forces in nature can be grouped in pairs inasmuch as corresponding to every individual force there is a second force which is exerted from the second point B on the first point A, and any two such corresponding forces are, according to the principle of equality of action and reaction (§ 29), equal to each other in magnitude and direction.

§ 77. We shall next deal in particular with the mechanics of point-systems which are at rest—that is, with statics—

and we shall fix our attention first on a system whose points always remain at fixed distances from one another in virtue of the forces that act between them, and which is therefore called a "rigid" body. A rigid body is of an absolutely invariable form in all its parts, but as a whole it can be set into motion by the smallest possible force. Perfectly rigid bodies do not occur in nature at all, but they are approximately realized by bodies that belong to the solid state of aggregation. The importance of rigid bodies for the theory does not depend on this circumstance alone, but rather on the fact that the mechanics of arbitrary point-systems may be reduced to the mechanics of rigid bodies (cf. § 130 below).

The problem to which we wish to devote the present chapter is the following. Let a stationary rigid body of arbitrary dimensions be given, on which forces of given magnitude and direction act at definite given points, the "points of action" (*Angriffspunkt*). We inquire into the condition under which the forces maintain equilibrium among themselves or, if this condition is not fulfilled, into the force or forces which must be applied in addition to establish equilibrium.

We solve this problem by reducing the given forces to as simple a form as possible, and in this process we proceed from the more special cases to the general case. The simplest case is that where only two forces are acting. In order that two forces which act on a rigid body should maintain equilibrium, it is obviously necessary that they should be equal in magnitude and opposite in direction. But this is not yet sufficient. For equilibrium it is also necessary that the line connecting the points of application A, B (Fig. 18) should coincide with the direction of the forces. For if the second force were to act at B' instead of at B there would be no equilibrium, and a rotation would occur.

This condition with regard to the direction of AB is in fact sufficient for equilibrium. The length of the distance AB and the form of the body do not matter at all.

The latter can be seen at once, if we imagine the body to be disposed perfectly symmetrically about the connecting line *AB* and also symmetrically with respect to the bisecting plane of the line *AB* (see figure). No one will doubt that equilibrium then occurs. Moreover, it is impossible for this equilibrium to be disturbed when the body is enlarged by the addition of arbitrary masses on which no forces are acting.

It immediately follows from this law that the physical meaning of a force which acts on a rigid body is no way changed if the point of application of the force is displaced by any arbitrary distance in the direction of the force. For, without causing disturbance, we may apply two equal and opposite forces *F* at any arbitrary point *C* of the body situated on the straight line *AB* (Fig. 18). Since the force which acts at *C* to the right and that which acts at *A* to the left maintain equilibrium these two forces may be omitted simultaneously, so that we are left with the force at *C* which acts towards the left instead of the force at *A* towards the left. But we cannot alter the point of application of a force in any direction other than that of the force (or in the opposite direction) without altering the physical meaning of the force. From this we see that besides the magnitude and direction also the point of application of a force plays a certain characteristic part and hence must always be specified if the force is to be regarded as completely known.

Of course the displacement of the point of application of a force is only allowed within the material of the body. But we may also make the displacement go beyond the limits of the body if we arrange that the point of application of the force remains rigidly connected with the body.

§ 78. If several forces act on a rigid body, whose directions all intersect at a point, it is easy to combine them into a single resultant—namely, by transferring the points of application to the common point of intersection, which, if it lies outside the body, must be regarded as being rigidly attached to it; and then we use § 24 to combine the forces that act at this point into a resultant *F* whose point of application may now again be displaced in the direction of *F* by any arbitrary amount.

As an example we shall consider the attraction which a rigid homogeneous sphere experiences, according to the Newtonian law of gravitation, as a result of the attraction of a material point *P* which lies outside it. The attraction is the resultant of the forces which the point *P* exerts on all the elements of mass of the sphere; the directions of all these forces pass through *P*. Hence we first transfer all these forces to *P* and there combine them into one resultant *F*. This is accomplished very easily if we reflect that the attractive force exerted by *P* on an element of mass of the sphere is equal and opposite to the attraction exerted by the element of mass on *P*. Accordingly, by § 33, *F* is equal and opposite to the force which the mass of the sphere, imagined as concentrated at the centre of the sphere, exerts on the point *P*. Now we may again transfer the point of application of *F* from *P* into the interior of the sphere, for example, to its centre. In this way we arrive at the theorem that the attraction which a rigid homogeneous sphere experiences on

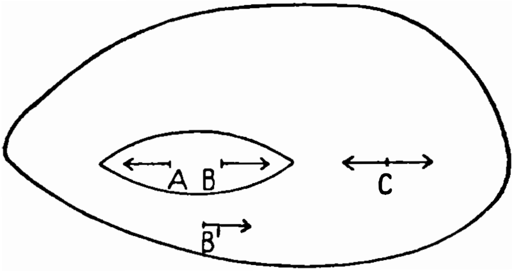

FIG. 18.

account of the action of a material point is of exactly the same amount as if its mass were all concentrated at its centre.

This simple theorem, however, holds only for a rigid sphere, and not, say, for a liquid sphere. For in the case of a liquid sphere the above line of argument, which operates with the transference of the points of application of the individual forces, which is legitimate only for rigid bodies, is not allowed. There is, in fact, an essential difference between the forces which a body exerts and the forces which it experiences. The former, if they act at a definite point, may be at once combined to form a single resultant, no matter how the body is constituted; the latter may be so combined only if the body is rigid.

Actually, the attraction which a liquid sphere experiences on account of gravitation cannot be combined into a single resultant force at all; rather, the sphere becomes deformed (cf. ebb and flow of the tides).

If the forces which act on the rigid body all lie in one plane, they may in general likewise be combined to form a single resultant. This may be done by selecting any two of them and forming the resultant at their point of intersection; the same process is then applied to the resultant and a third force, and so forth until the last force has been used. There is an exception, however, in the case of parallel forces, which we must therefore treat specially.

If among a number of forces there are even only two whose directions do not lie in one plane, the process which has here been described for compounding the forces becomes illusory, because the force cannot be transferred to a common point of application. To solve this most general case it is therefore necessary to extend the theory. We shall first consider the case of parallel forces.

§ 79. **Parallel Forces.** We choose as the plane of our diagram the plane defined by the points of application *A* and *B* and the directions of the two given forces F_1 and F_2 which act in the same sense (Fig. 19). Since the directions of F_1 and F_2 do not intersect we introduce two equal and opposite additional forces *K* which act at *A* and *B* in the direction *AB* and exactly cancel each another. Then at *A* the force F_1 combines with *K* to form G_1, and at *B* the force F_2 combines with *K* to form G_2, and now we can easily construct the resultant of G_1 and G_2 by transferring the points of application *A* and *B* to the point of intersection *C* (see Fig.). The forces G_1 and G_2 are compounded most easily by first resolving G_1 and G_2 again into their components F_1 and *K* and F_2 and *K*, respectively; this is equivalent simply to transferring the force-parallelograms from *A* and *B* to *C*. The two forces *K* then again cancel and we get the resultant:

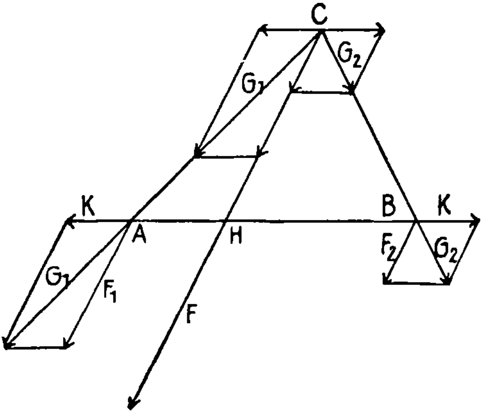

FIG 19.

$$F = F_1 + F_2 \,. \quad . \quad . \quad . \quad . \qquad (281)$$

which is parallel to the forces F_1 and F_2 and which acts at *C* or at any point in its own direction, for example at its point of intersection *H* with the straight line *AB*. This point *H* is distinguished from all other points of the straight line CH in that its position depends only on the magnitudes and points of application of the forces F_1 and F_2, but not on their direction.

We get this result if we reflect that the triangle formed by the forces F_1, K, G_1 (one-half of the parallelogram of forces) is similar to the triangle ACH, and hence :

$$AH : HC = K : F_1.$$

In the same way :

$$BH : HC = K : F_2.$$

Consequently :

$$F_1 \cdot AH = F_2 \cdot BH \quad . \quad . \quad . \quad . \quad (281a)$$

or :

$$AH = \frac{F_2}{F_1 + F_2} \cdot AB \quad . \quad . \quad . \quad . \quad (282)$$

Hence if we keep the magnitude of the parallel forces F_1 and F_2 constant, but rotate them about their points of application A and B, the resultant F also rotates about its point of application H and remains unaltered in magnitude.

When $F_2 = 0$, H coincides with A, and when $F_2 = F_1$, H lies midway between A and B, as is natural. But in all cases H lies between A and B.

To deal with the case of any arbitrary number of parallel forces we shall introduce the analytical method of treatment. Let x_1, y_1, z_1 be the co-ordinates of A, and x_2, y_2, z_2 the co-ordinates of B. Then the equation to the straight line AB is :

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Moreover, the co-ordinates x_0, y_0, z_0 of the point H are determined by the fact that they satisfy the above equation and also the equation (282). Thus :

$$\frac{x_0 - x_1}{x_2 - x_1} = \frac{y_0 - y_1}{y_2 - y_1} = \frac{z_0 - z_1}{z_2 - z_1} = \frac{AH}{AB} = \frac{F_2}{F_1 + F_2}.$$

Hence in view of (281) it follows that :

$$\left.\begin{aligned} x_0(F_1+F_2) &= x_0F = x_1F_1+x_2F_2 \\ y_0(F_1+F_2) &= y_0F = y_1F_1+y_2F_2 \\ z_0(F_1+F_2) &= z_0F = z_1F_1+z_2F_2 \end{aligned}\right\} \quad . \qquad (283)$$

That is, the product of the resultant force F and one co-ordinate of its point of application is equal to the sum of the products of the components and the corresponding co-ordinate of its point of application.

It is easy to generalize this result for any arbitrary number of parallel forces.

For example, if we have three forces F_1, F_2, F_3 we first imagine F_1 and F_2 to be combined into a resultant F', whose magnitude is F_1 and F_2 and whose point of application x'_0, y'_0, z'_0 is given by (283).

Then the required force F is the resultant of F' and F_3, and hence, by (281), its magnitude is :

$$F = F' + F_3 = F_1 + F_2 + F_3,$$

and, by (283), its point of application is determined by the fact that the product x_0F is equal to the sum of $x_3 F_3$ and $x'_0 F'$.

But by (283) the latter product is again equal to :

$$x_1F_1 + x_2F_2.$$

Consequently :

$$x_0F = x_1F_1 + x_2F_2 + x_3F_3,$$

and similar expressions hold in the co-ordinates y and z.

Hence in the case of any arbitrary number of parallel forces F_1, F_2, F_3, . . . all acting in the same sense at the points x_1, y_1, z_1, x_2, y_2, z_2, . . . the magnitude of the resultant force is given by :

$$F = \Sigma F_1 \quad . \quad . \quad . \quad . \quad . \qquad (284)$$

and the point at which the resultant acts, namely, x_0, y_0, z_0, is given by :

$$x_0F = \Sigma x_1F_1,\ y_0F = \Sigma y_1F_1,\ z_0F = \Sigma z_1F_1 \quad . \qquad (285)$$

and the point of application may be displaced by any arbitrary amount in the direction of F.

§ 80. **Application to Gravity.** As a particularly important application of the theorems which have just been deduced, we consider the question of the resultant of all the forces which the earth exerts on a rigid body according to the law of gravitation. The earth acts on an element of mass m_1 of the body with the force $m_1 g$ in the direction of the earth's centre (§ 34). So long as the dimensions of the body remain vanishingly small compared with its distance from the earth's centre, the forces of attraction on all the individual elements of mass $m_1, m_2, \ldots$ of the body may be regarded as parallel and hence combine into a single resultant F in the same direction and of magnitude given according to (284) by:

$$F = g \cdot \Sigma m_1 \quad . \quad . \quad . \quad . \quad . \qquad (286)$$

which acts at the point (x_0, y_0, z_0), where by (285):

$$x_0 \Sigma m_1 = \Sigma m_1 x_1, \ldots \quad . \quad . \quad . \qquad (287)$$

The point x_0, y_0, z_0 determined by these equations is called the *centre of gravity* of the body. Its position depends not only on the direction of the force of gravity and on the magnitude of the acceleration g due to gravity, but also on the relative positions of the elements of mass in the body. Hence it has a far greater importance than that of being the point of application of the resultant gravitational force and is more correctly called the *centre of mass* of the body.

It is often found convenient to speak of the centre of gravity of elements of mass even when they are not rigidly connected with one another; for example, we speak of the centre of gravity of a system of freely movable points of masses, $m_1, m_2, \ldots$, where we regard this centre of gravity to be defined at every moment by the equations (287). In this case the significance of the centre of gravity as the point of application of the resultant gravitational force becomes altogether void of sense.

If we wish to find the centre of gravity of a *system* of bodies, it is often expedient to perform the summation in (287) not directly over all the elements of mass of all the bodies, but first for each body individually and then, by imagining the mass of the body to be concentrated in this centre of gravity of the body, to determine the centre of gravity again for the new point-masses so obtained.

The fact that this method of procedure always leads to the correct result is seen most simply by supposing all the bodies to be rigidly connected together and to have weight. For the resultant force of gravitation of the whole rigid system will certainly come out correctly if we first form the resultant force of gravity for every body individually and then again combine the forces so obtained into a single resultant.

If the mass of a body is distributed continuously in space the element of volume dV contains the mass kdV (§ 31), where the density k may depend on the co-ordinates x, y, z; then the sums become replaced by integrals. From (287) we get in this case for the position of the centre of gravity:

$$x_0\int kdV = \int kxdV, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (288)$$

If, in particular, the body is homogeneous—that is, if k is constant—k cancels out entirely, and we have:

$$x_0\int dV = \int xdV. \quad . \quad . \quad . \quad . \quad . \qquad (289)$$

It is in this sense that we also speak of the centre of gravity of a volume, and likewise of the centre of gravity of a surface or of a line, by imagining the geometrical configuration in question to be uniformly covered with mass, whose density then cancels out in each case.

Let us, for example, calculate the position of the centre of gravity for the surface of a circular sector of radius r and angle of aperture α.

We take the centre of the circle as our origin of co-ordinates and the bisector of the angle α as the x-axis.

We then easily get in the manner of (289), using ρ and ϕ as polar co-ordinates:

$$x_0\int\int\rho d\rho d\phi = \int\int\rho\cos\phi\cdot\rho d\rho d\phi$$

and:

$$y_0\int\int\rho d\rho d\phi = \int\int\rho\sin\phi\cdot\rho d\rho d\phi$$

with the limits 0 and r for ρ, and $-\frac{\alpha}{2}$ and $+\frac{\alpha}{2}$ for ϕ.

Hence:

$$x_0 = \frac{4}{3\alpha}\sin\frac{\alpha}{2}\cdot r,\ y_0 = 0 \quad . \qquad (290)$$

For $\alpha = 2\pi$ we have the complete area of the circle; for this case $x_0 = 0$. For $\alpha = 0$, however, we get $x_0 = \frac{2}{3}r$, which corresponds to the centre of gravity of an infinitely narrow triangular area whose base is at a distance r from its vertex.

We must be careful to distinguish between the centre of gravity of a triangular area and the centre of gravity of the periphery of the triangle. The latter can be found most easily by means of the theorem deduced above, by first finding the centres of gravity of the separate sides (namely, their mid-points) and then supposing each of these points to have the mass of the whole side (measured by its length).

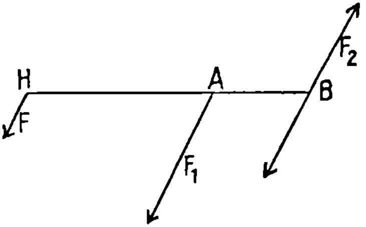

FIG. 20.

§ 81. **Anti-parallel Forces.** Let us consider two forces F_1 and F_2 which act at the points A and B, but in opposite senses; such forces are said to be "anti-parallel." Suppose that $F_1 > F_2$ (Fig. 20).

We then get the resultant force most simply as follows. We resolve the greater force F_1 into two parallel forces that act in the same sense, one of these acting at B and being equal and opposite to F_2, whereas the other, F, acts at a point H on the other side of A. This is always possible if we arrange that F_1 is the resultant of these two forces—that is, so that by (281):

$$F_1 = F + F_2 \quad . \qquad . \quad (291)$$

and by (281*a*):

$$F \, . \, HA = F_2 \, . \, BA \quad . \quad . \quad . \quad . \quad (292)$$

We next substitute for the force F_1 its two components F and F_2. Then the two forces F_2 at B cancel out and we are left with the force F, whose magnitude is given, by (291), as:

$$F = F_1 - F_2 \, . \quad . \quad . \quad . \quad . \quad (293)$$

and whose direction is the same as that of the greater force F_1. The point H at which it acts lies outside the line AB on the side of the greater force F_1, its distance from the point of application A being given, by (292), as:

$$AH = AB \cdot \frac{F_2}{F} \quad . \quad . \quad . \qquad (294)$$

The equations (293) and (294) may also be regarded as generalizations of the equations (281) and (282), which were derived for parallel forces, since they arise from the latter if F_2 is taken as negative. Then the negative value of AH indicates that H lies on the side of A remote from the point B.

The more F_2 approaches F_1 in magnitude, the further H moves off, and when $F_2 = F_1$ our method of determining the resultant force becomes illusory. Two equal anti-parallel forces cannot be combined into a single resultant force at all; they form a special class of forces called "couples."

If we have any arbitrary number of parallel and anti-parallel forces acting on a rigid body they can in general be combined into a single resultant. For a simple consideration along the lines of the result obtained for two anti-parallel forces shows that the formulæ (284) and (285) for the resultant of a system of parallel forces are still applicable even if some of the forces act in the opposing sense. We need only introduce the latter force into the equations with a minus sign. Then the sign of the algebraic sum of all the forces ΣF_1 gives the direction of the resultant, whereas the value of ΣF_1 gives its magnitude.

There is an exception, however, in the case $\Sigma F_1 = 0$. For here the equations (285) that serve to determine the point of action x_0, y_0, z_0 of the resultant lose their meaning, and the whole system of forces reduces either to a couple or it maintains equilibrium.

Which of these two cases occurs is decided by the following considerations. We first combine all the forces that act in the one direction F'_1, F'_2, F'_3, . . . into a resultant F', and then all those that act in the opposite direction, F''_1, F''_2, F''_3, . . . (taken positively) into their resultant F''.

By our assumption we then have :

$$\Sigma F'_1 = \Sigma F''_1 \quad . \quad . \quad . \quad . \quad . \qquad (295)$$

Further :

$$\left.\begin{aligned} x'_0 \Sigma F'_1 &= \Sigma x'_1 F'_1 \ldots \\ \text{and : } x''_0 \Sigma F''_1 &= \Sigma x''_1 F''_1 \ldots \end{aligned}\right\} \quad . \quad . \qquad (296)$$

We next investigate whether the line connecting the points of application of the two equal anti-parallel resultants F' and F'' coincides with their direction or not.

In the former case we have equilibrium ; in the latter case we have a couple. For equilibrium the following condition must be satisfied :

$$(x''_0 - x'_0) : (y''_0 - y'_0) : (z''_0 - z'_0) = \cos\alpha : \cos\beta : \cos\gamma$$

where α, β, γ are the direction-angles of the forces.

If we substitute the values (296) in this expression we get, if we take (295) into account and introduce the symbol F (positive or negative) for the magnitude and direction of a force, that the necessary and sufficient condition for the equilibrium of a system of parallel and anti-parallel forces is :

$$\left.\begin{aligned} \Sigma F_1 &= 0 \\ \text{and : } \quad \Sigma x_1 F_1 : \Sigma y_1 F_1 : \Sigma z_1 F_1 &= \cos\alpha : \cos\beta : \cos\gamma \end{aligned}\right\} \qquad (297)$$

For example, if the forces are parallel or anti-parallel to the z-axis, then $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}, \gamma = 0$, and the conditions for equilibrium become :

$$\Sigma F_1 = 0, \ \Sigma x_1 F_1 = 0, \ \Sigma y_1 F_1 = 0$$

The z-co-ordinates of the points of application do not enter here at all—as is natural, since every force can be displaced arbitrarily in the direction of the z-axis.

§ 82. Let us now deal with couples more particularly. We choose as the plane of our diagram the plane of a couple which consists of two anti-parallel forces F and we displace the point of application of the one force F so far in its own direction that the line connecting the two points of application AB is perpendicular to F. Then AB is called the " arm " of the couple.

It is easy to see that a couple may be displaced as far as we please in the direction of one of the forces without altering its physical meaning, for example, to $A'B'$ (Fig. 21). For what is allowed by § 77 for any single force must also be possible for the two forces together.

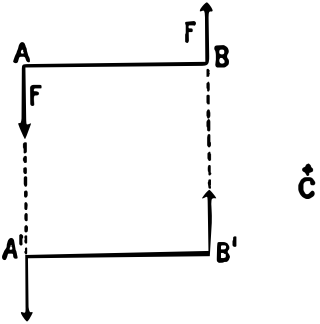

FIG. 21.

Misgivings which may arise about this theorem and the way in which they are disposed of may best be dealt with in the form of a little dialogue.

" How can the couples AB and $A'B'$ be equivalent, since a rotation of the body about the middle of AB is not the same as a rotation of the body about the middle of $A'B'$? "

The two rotations mentioned are certainly not identical. But it has not been asserted, and it is not at all true, that a couple causes a rotation of the body about the mid-point of its arm.

"But by § 76 every point moves according to the resultant of the force that acts on it. Consequently, if the couple acts at A and B the points A and B, which are initially at rest, move in the direction of their forces F, and since the forces are equal, this motion is a rotation about the centre of AB."

By § 76 every point moves according to the resultant of the forces which are exerted on it by all the other forces in the universe.

Now in the present case we have not two isolated points A and B, but a rigid body to which the points A and B belong. Thus they are under the influence of the forces which are exerted on them by the other points of the body, particularly by those in their immediate neighbourhood which cause the body to be rigid. These internal forces must also be taken into account when we are dealing with the motion of the points A and B, and not only the forces F.

"But the internal forces of the body are unable to set the body into motion; they maintain equilibrium and may therefore be omitted."

It is true that the internal forces of a rigid body maintain equilibrium if they are all combined together to form a resultant. This follows from § 76 by the principle of the equality of action and reaction (Newton's third law). But here we are dealing, not with the resultant of all the internal forces of the body, but with the resultant of those internal forces which act on the point A (or on the point B).

The fact that these forces are not always in equilibrium is most easily recognized by considering any other arbitrary point C of the body (Fig. 21).

If the body is set into motion by the couple the point C will also begin to move. What force causes its motion? Only the resultant of the internal forces that act on it, for they are the only forces to which it is subject. Just as much as the internal forces act on C with a finite resultant, so they may also in general act on A or B, and it is this resultant combined with F which determines the motion of A and B. So we see that a couple does not necessarily rotate the body about the mid-point of its arm and that our misgivings about the complete equivalence of the couples AB and $A'B'$ are unfounded.

"But how does the body actually move under the influence of a couple if it does not rotate about the mid-point of its arm?"

This question cannot be fully answered at this stage. The complete answer, which is unambiguous, will be given below (§ 149).

But a couple may also be displaced by an arbitrary amount in the direction of its arm AB without its significance being affected.

For if we apply two forces F, equal and opposite to the original forces, at each of two points A' and B' situated on the straight line AB, and such that $A'B' = AB$, then these forces do not disturb the system of forces at all, since they cancel out in pairs (Fig. 22). The force F at A combines with the force F at B', which is parallel to it, to form a parallel resultant $2F$ which acts at the mid-point O of AB'. In the same way, the opposite parallel forces at B and A' combine to form the resultant $2F$, which is parallel to each of them and which acts at the mid-point of BA'—that is, also at O. Thus these two resultants cancel out and we are left with only the couple at $A'B'$, which is nothing else than the displaced original couple.

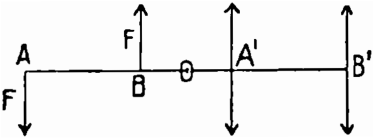

FIG. 22.

Furthermore, the arm of the couple may also be rotated by an arbitrary amount about its mid-point O in the plane of the couple—that is, of the diagram. To show this, we again suppose two equal and opposite forces F to be applied perpendicularly to $A'B'$ when the arm $A'B'$ has been rotated through an arbitrary angle. These two forces do not disturb the system of forces (Fig. 23), and on the one hand combine the original force at A with the force at A' which points towards it, and on the other hand the original force at B with the force at B' which points towards it, to form a single resultant in each case; we displace the two components as far as the point of intersection in each case (see Fig.).

Since the forces are equal, the resultants lie in the direction of the bisector of the angle between the two arms,

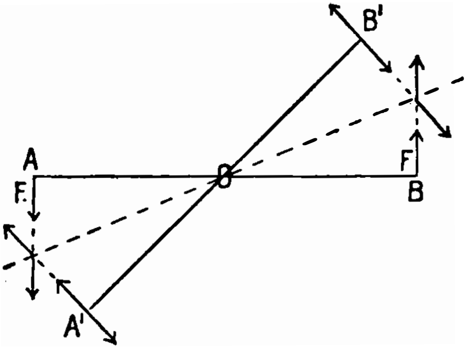

FIG. 23.

and since they are equal and opposite, they cancel out. Hence we are again left with the original couple, but with its arm rotated. If the arm is rotated through the angle π we again get the original couple.

It is clear then that by successively displacing the couple parallel to itself and rotating it we can transport the couple to any position in its own plane without altering its physical meaning.

But we may go still further. The couple may also be transferred to any other parallel plane. To show this we suppose that the plane of the couple is horizontal (Fig. 24 which depicts the plane in perspective), the force F at A pointing towards the reader, the force at B in the opposite direction. We then again apply two equal and opposite forces F at the points A' and B' at the same heights vertically above A and B; these forces do not disturb the system.

The force at A, which points towards the observer, when combined with the parallel force at B' acting in the same direction, gives a resultant $2F$, which is equal and opposite to the resultant of the force at B acting away from the observer and the force at A' in the same direction. These resultants also act at the same point, for the mid-point of the distance AB' is also the mid-point of the distance BA'. Consequently we are left with only the original couple, but now displaced into the upper plane and capable of being displaced in any arbitrary way in that plane.

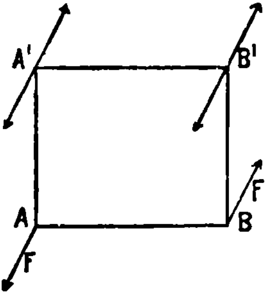

FIG. 24.

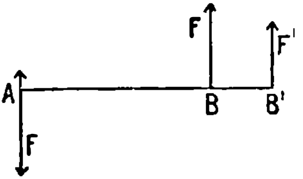

FIG. 25.

Finally, the length of the arm AB may also be altered by an arbitrary amount without the physical meaning of the couple being affected. For if we resolve the force F which acts at B into two parallel components, of which one, F', acts at an arbitrary point B' of the straight line AB, whereas the other, $F - F'$, is assumed to act at A (Fig. 25), then we may replace the force at B by its two components, if we have, by (281a):

$$F' \,.\, B'B = (F - F') \,.\, AB \quad . \quad . \quad . \quad (298)$$

At A there then remains a force $F - (F - F') = F'$, and at B an equal anti-parallel force F'—that is, we have a couple whose force is F' and whose arm is AB', where, by (298):

$$F' \,.\, AB' = F \,.\, AB \quad . \quad . \quad . \quad (299)$$

That is, the product of the force and the length of arm are exactly the same for the new couple as for the old. If we call this product the *moment* of the couple, we have the theorem that two couples situated in the same plane or in parallel planes and having equal moments are identical.

§ 83. Having become acquainted with the properties of transformation of couples, we may now also answer the question as to what actually characterizes a couple at all. According to the above theorems a couple is obviously determined first by its moment, secondly by the direction of its plane, and thirdly by its sense. In Fig. 26 we see two couples which have the same moment and lie in the same plane but which are nevertheless not identical. For they maintain equilibrium—that is, they are exactly opposite to each other. This compels us to assign a sense (of rotation) to a couple. This sense suggests itself if we fix our attention on the rotation which is indicated by the directions of the two force arrows. To be able to define the sense of rotation uniquely we shall ascribe to every rotation a *definitely directed axis of rotation*, and we do this once and for all by making the convention that a rotation of a co-ordinate system about its origin, such that the positive x-axis moves towards the positive y-axis, has the positive z-axis as its axis of rotation. The reverse rotation has the negative z-axis as its axis of rotation. Since we always use right-handed co-ordinate systems (§ 16), this convention is equivalent to the following: the axis of rotation of the hands of a clock is from the observer towards the dial; or the axis of rotation of a cork-screw which is being twisted into a cork is the direction in which the cork-screw as a whole moves in the fixed cork; or the axis of rotation of the earth is in the direction from the South Pole to the North Pole.

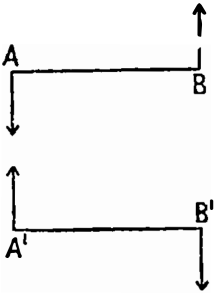

FIG. 26.

Accordingly, the axis of the couple *AB* in Fig. 26 is in the direction pointing from the plane of the diagram to the observer, that of the couple $A'B'$ is from the observer to the plane of the diagram.

This convention enables us to represent a couple, just like a force, by means of a simple geometrical symbol—namely, by means of a directed distance whose length denotes the moment N and whose direction denotes the axis of the couple. But in order to prevent confusion with the symbol for a force we shall indicate the direction of the axis by means of a *double* arrow-head.

Accordingly, the vertical (double) arrow in Fig. 27 denotes that couple whose moment N is equal to the length of the arrow and whose (horizontal) plane is perpendicular to its direction, and whose axis is in the direction of the double arrow-head. (The corresponding forces are shown in perspective by means of dotted lines.) This symbol may be displaced in any way, even laterally, so long as it remains of the same size and parallel to itself. The "point of application" of a couple, then, has, in contradistinction to the point of application of a force, not the slightest physical meaning.

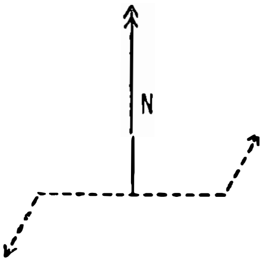

FIG. 27.

§ 84. **Composition of Couples.** Through introducing the above described symbol for a couple we are able to formulate very simply the laws according to which couples are compounded.

Let us first consider a system of any arbitrary number of couples N_1, N_2, N_3, . . . with parallel axes. These may all be brought to lie in one plane and, when in this plane, to act at a common arm. At the one end of the arm the forces all act perpendicularly to the arm, and so are compounded simply by addition to form a resultant, to which there is a corresponding anti-parallel resultant at the other extreme of the arm. Thus the result is a single couple, whose moment is represented by the product of the length of arm and the sum of the individual forces—that is, by the sum of all the moments N_1, N_2, N_3, . . .—and whose axis is the common axis. If anti-parallel couples are also present we need only take their moments as negative; we then obtain the sense and magnitude of the resultant moment by adding all the moments algebraically.

We next consider two couples N_1 and N_2 whose axes form any arbitrary angle. We again combine the couples at a common arm AB on the straight line in which the planes of the two couples intersect. We describe the plane of Fig. 28 through the point A and perpendicularly to the arm, so that the other extreme B of the arm lies behind the plane of the diagram. Then the forces F_1 and F_2, which act at A, lie in the plane of the diagram, and likewise the axes N_1 and N_2, but at right angles to the forces, in the sense indicated. According to the parallelogram law, the forces F_1 and F_2 combine to form the resultant F, to which there is an equal anti-parallel resultant at B. The result is a couple of moment N, the direction of the axis of N being perpendicular to F, and the ratio N to F having the value:

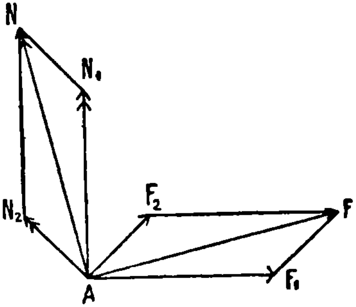

FIG. 28.

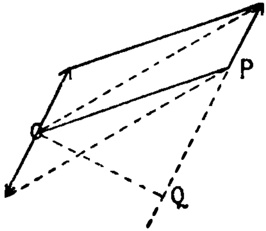

FIG. 29.

$$\frac{N}{F} = \frac{N_1}{F_1} = \frac{N_2}{F_2} = AB.$$

The quadrilateral formed by the distances N is a parallelogram; for it is similar to the parallelogram of the forces F, and is rotated through a right angle with respect to them. Hence we have the theorem that if we characterize any arbitrary couples by means of their symbols, they are compounded and resolved, like forces, according to the parallelogram law. In other words, *couples behave like vectors.* This induces us to denote couples by the letter $\mathbf{N}$ in clarendon type. The absolute (or numerical) value of the vector $\mathbf{N}$ is the moment N, and its direction is the axis of the couple. Couples $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, . . . no matter how great their number, which act on a rigid body are compounded quite generally by vectorial addition to form the resultant couple:

$$\mathbf{N} = \Sigma \mathbf{N}_1 \quad . \quad . \quad . \quad . \quad . \qquad (300)$$

or, expressed in terms of the components:

$$\left.\begin{aligned} N \cos\lambda &= \Sigma N_1 \cos\lambda_1 = \Sigma N_{x1} \\ N \cos\mu &= \Sigma N_1 \cos\mu_1 = \Sigma N_{y1} \\ N \cos\nu &= \Sigma N_1 \cos\nu_1 = \Sigma N_{z1} \end{aligned}\right\} \quad . \quad . \qquad (301)$$

where λ, μ, ν denote the direction-angles of an axis.

§ 85. **Composition of Arbitrary Forces.** The problem the solution of which we had to postpone in § 78 because we were unable there to compound two forces whose directions did not intersect may now be taken up again, since we have learned how to compound couples quite generally and since, as we shall presently show, it is possible to displace the point of application of a force quite arbitrarily by introducing the appropriate couples.

For if we have a force which acts at the point P (Fig. 29) we may introduce at any arbitrary point O of the rigid body two equal forces F, one parallel and one anti-parallel, which mutually cancel.

We then get as a result first the force F displaced to the point O and secondly the couple composed of the original force F and the anti-parallel force at O.

Let us calculate its moment and the direction of its axis. The value of the moment is the product of F and the length OQ of the perpendicular dropped from O on the direction of the original force that passes through P. It is represented by the surface of the parallelogram formed by the sides OP and F.

This quantity is therefore called the " (statical) moment of the force F acting at P with respect to the point O."

If O lies in the direction of the force PF, the moment vanishes. The axis of the moment is perpendicular to the plane OPF and is directed from the diagram towards the observer in Fig. 29.

We now see immediately how any arbitrary forces acting at any arbitrary points may be compounded: the forces are all displaced in the manner just described to a common point of application—for example so that they all act at the origin of co-ordinates; here they are all combined into one resultant. Besides this resultant we then also have the couples which result from the displacement of the forces, and by § 84 these couples are likewise all combined to form a single couple.

§ 86. To apply this idea in actual calculations we first consider the displacement of a force $\boldsymbol{F}$ acting at the point $\boldsymbol{r}$ to the origin O, and calculate the three components of the couple which results from this process. For it is these components which we require afterwards in compounding the different couples. We find it expedient to use Fig. 3 (§ 17) for this purpose, and we consider individually the three components $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, $\boldsymbol{F}_z$ of the force $\boldsymbol{F}$ that acts at the point P. Let us first take the component $\boldsymbol{F}_z$. This component may immediately be transferred from P to the point B in the xy-plane, because BP lies in the direction of the force. But if we now displace $\boldsymbol{F}_z$ further from B to the point A on the x-axis, we get in the process a couple parallel to the yz-plane, of moment $AB\,.\,\boldsymbol{F}_z = y\boldsymbol{F}_z$, whose axis is the positive x-axis. If, finally, we displace $\boldsymbol{F}_z$ from A to O we get a couple which is parallel to the xz-plane; it has the moment $AO\,.\,\boldsymbol{F}_z = x\boldsymbol{F}_z$ and its axis is the negative y-axis.

The displacement of the other two force components $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$ of P to O may be carried out simply by a cyclic exchange of letters in the results already obtained. Accordingly the displacement of $\boldsymbol{F}_x$ leads to two couples of moments $z\boldsymbol{F}_x$ and $y\boldsymbol{F}_x$, whose axes are the positive y-axis and the negative z-axis; and the displacement of $\boldsymbol{F}_y$ leads to two couples of moments $x\boldsymbol{F}_y$ and $z\boldsymbol{F}_y$, whose axes are the positive z-axis and the negative x-axis.

By (301) these six couples combine to form a single couple $\boldsymbol{N}$, where

$$\left.\begin{aligned} \boldsymbol{N}_x &= y\boldsymbol{F}_z - z\boldsymbol{F}_y \\ \boldsymbol{N}_y &= z\boldsymbol{F}_x - x\boldsymbol{F}_z \\ \boldsymbol{N}_z &= x\boldsymbol{F}_y - y\boldsymbol{F}_x \end{aligned}\right\} \quad . \tag{302}$$

§ 87. A vector $\boldsymbol{N}$, which is composed of the two vectors $\boldsymbol{r}$ and $\boldsymbol{F}$ according to (302) is called the " vector product " (or " external product ") of $\boldsymbol{r}$ and $\boldsymbol{F}$ to distinguish it from the " scalar product " (or " inner product ") $\boldsymbol{r} . \boldsymbol{F} = x\boldsymbol{F}_x + y\boldsymbol{F}_y + z\boldsymbol{F}_z$ (§ 47), and is designated by :

$$\boldsymbol{N} = [\boldsymbol{r}, \boldsymbol{F}] = -[\boldsymbol{F}, \boldsymbol{r}] \quad . \quad . \quad . \tag{303}$$

By § 85 the absolute value of the vector product of $\boldsymbol{r}$ and $\boldsymbol{F}$ is equal to the area of the parallelogram formed by the vectors $\boldsymbol{r}$ and $\boldsymbol{F}$, and its direction is the normal to this parallelogram, in such a way that the directions $\boldsymbol{N}$, $\boldsymbol{r}$, $\boldsymbol{F}$ or $\boldsymbol{r}$, $\boldsymbol{F}$, $\boldsymbol{N}$ or $\boldsymbol{F}$, $\boldsymbol{N}$, $\boldsymbol{r}$ form a right-handed system, which, moreover, is right-angled if $\boldsymbol{r} \perp \boldsymbol{F}$.

These theorems may of course be directly derived from (302): the fact that $\boldsymbol{N} \perp \boldsymbol{r}$ and $\perp \boldsymbol{F}$ may be seen by multiplying the individual equations (302) with the components of $\boldsymbol{r}$ or with the components of $\boldsymbol{F}$ and subsequently adding. Also, for the square of the absolute value of $\boldsymbol{N}$ we get by squaring and adding (302) :

$$\begin{aligned} &(y\boldsymbol{F}_z - z\boldsymbol{F}_y)^2 + (z\boldsymbol{F}_x - x\boldsymbol{F}_z)^2 + (x\boldsymbol{F}_y - y\boldsymbol{F}_x)^2 \\ &\quad = (x^2+y^2+z^2)(\boldsymbol{F}_x{}^2+\boldsymbol{F}_y{}^2+\boldsymbol{F}_z{}^2) - (x\boldsymbol{F}_x+y\boldsymbol{F}_y+z\boldsymbol{F}_z)^2 \qquad (304) \\ &\quad = \boldsymbol{r}^2\boldsymbol{F}^2 - \boldsymbol{r}^2\boldsymbol{F}^2\cos^2(\boldsymbol{r}, \boldsymbol{F}) \\ &\quad = \boldsymbol{r}^2\boldsymbol{F}^2\sin^2(\boldsymbol{r}, \boldsymbol{F}), \qquad (305) \end{aligned}$$

that is, the square of the parallelogram formed by $\boldsymbol{r}$ and $\boldsymbol{F}$.

The quantities $\boldsymbol{N}_x$, $\boldsymbol{N}_y$, $\boldsymbol{N}_z$, defined by (302) are also called the (statical) moments, with respect to the three co-ordinate axes, of the force $\boldsymbol{F}$ which acts at P. Hence the moment of a force with respect to any straight line in space is equal to the product of the component of the force perpendicular to this line and the distance from the force to the straight line. We confirm that this theorem is correct by reflecting that the moment $\boldsymbol{N}_z$ of the force $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, $\boldsymbol{F}_z$ acting at the point (x, y, z) with respect to the z-axis is, by (302), equal to the moment of the force $(\boldsymbol{F}_x, \boldsymbol{F}_y, 0)$ acting at the point $(x, y, 0)$ with respect to the origin of co-ordinates (§ 85).

§ 88. Taking into consideration the idea described at the conclusion of § 85, we may now write down directly the result of compounding any arbitrary forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ . . . acting at the points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, ... The result is :

$$\left.\begin{aligned}\boldsymbol{F} &= \Sigma\boldsymbol{F}_1\\ \boldsymbol{N} &= \Sigma[\boldsymbol{r}_1 . \boldsymbol{F}_1]\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (306)$$

Or, in words, if any forces whatsoever act on a rigid body, they may always be compounded into a single force $\boldsymbol{F}$ which acts at the origin of co-ordinates and a single couple $\boldsymbol{N}$, where $\boldsymbol{F}$ and $\boldsymbol{N}$ are compounded from the given forces in accordance with (306). These equations (306) contain all the theorems hitherto derived, including those dealing with parallel and anti-parallel systems of forces, as special cases. If we consider that of the two forces of the resultant couple $\boldsymbol{N}$ the one force may be supposed to act at the origin of co-ordinates and may there be combined with the resultant force $\boldsymbol{F}$ to form a new resultant, it becomes clear that the most general case of a system of forces acting on a rigid body may also be reduced to two forces.

For a system of forces acting on a rigid body to maintain equilibrium it is sufficient but also necessary that both the resultant force $\boldsymbol{F}$ as well as the resultant couple $\boldsymbol{N}$ should vanish. By (306) this gives :

$$\Sigma\boldsymbol{F}_1 = 0, \ \Sigma[\boldsymbol{r}_1, \boldsymbol{F}_1] = 0 \quad . \quad . \quad . \quad (306a)$$

So we have six equations of condition between the components of the forces and the co-ordinates of the points of application.

§ 89. A certain arbitrariness occurs in our method of reducing any system of forces whatsoever to a single resultant and a single couple in that the point of application of the resultant was chosen at random. The question arises whether and how the result becomes altered if all the forces, instead of being displaced to the point O, are displaced to a different point O_0. In particular, it would be of interest to investigate whether it is not possible by suitably choosing the point O_0 to get a specially simple result for the reduction of the system of forces. This answer to this question may be found most conveniently by displacing the resultant $\boldsymbol{F}$ and the couple $\boldsymbol{N}$, which represent the whole system of forces, directly from the point of application O to the point of application O_0. The displacement of $\boldsymbol{F}$ then gives rise to a new couple $\boldsymbol{N}'$, whose axis is perpendicular to $\boldsymbol{F}$ and to OO_0, and which combines with $\boldsymbol{N}$ to form a single couple $\boldsymbol{N}_0$.

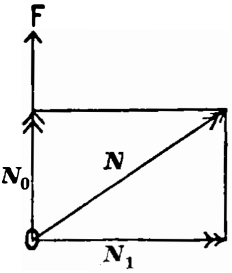

FIG. 30.

Thus we finally obtain at O_0 the resultant $\boldsymbol{F}$ and the couple $\boldsymbol{N}_0$; from this we see at once that the resultant force $\boldsymbol{F}$ of a system of forces is quite independent of the position of its point of application O_0, whereas on the other hand the associated couple *does* depend on O_0. Can we choose O_0 so that $\boldsymbol{N}'$ and $\boldsymbol{N}$ exactly cancel—that is, so that $\boldsymbol{N}_0 = 0$? In general, it is clear that we cannot do so. For then $\boldsymbol{N}'$ would have to be equal and opposite to $\boldsymbol{N}$, whereas actually $\boldsymbol{N}'$ is restricted by the condition $\boldsymbol{N}' \perp \boldsymbol{F}$, which does not hold for $\boldsymbol{N}$.

But the following simplification may always be achieved. If we resolve $\boldsymbol{N}$ into a component $\boldsymbol{N}_0$ parallel to $\boldsymbol{F}$ and a component $\boldsymbol{N}_1$ perpendicular to $\boldsymbol{F}$ (Fig. 30), we may choose the new point of application O_0 so that the couple $\boldsymbol{N}'$ which results from the displacement of $\boldsymbol{F}$ to O_0 becomes equal to $-\boldsymbol{N}_1$.

We need only choose O_0 to lie on the receding normal of the plane formed by $\boldsymbol{F}$ and $\boldsymbol{N}$ (the plane of the diagram) and make the distance $OO_0 = \frac{N_1}{F}$.

Then only the force $\boldsymbol{F}$ and the couple $\boldsymbol{N}_0$ remain at O_0, and we obtain the theorem that *every system of forces acting on a rigid body may be reduced to a force and a couple whose axis lies in the direction of the force.*

To obtain a convenient survey of the conditions that hold in the most general case, we imagine that at every point O of the body we construct the resultant $\boldsymbol{F}$ that acts at it and the corresponding couple $\boldsymbol{N}$. It is then clearly sufficient to consider all points O of a plane perpendicular to $\boldsymbol{F}$; for $\boldsymbol{F}$ and $\boldsymbol{N}$ are the same on every straight line parallel to $\boldsymbol{F}$. We take as such a plane the plane of the diagram in Fig. 31. Let O_0 be that point of the plane at which the axis of the corresponding couple $\boldsymbol{N}_0$ coincides with $\boldsymbol{F}$, and hence, like $\boldsymbol{F}$, is perpendicular to the plane. We suppose the force $\boldsymbol{F}$ to be directed towards the reader. If we pass over to another point O of the plane we get as a result of displacing the force $\boldsymbol{F}$ from O_0 to O a couple of moment $N' = O_0O \,.\, F$, whose axis lies in the plane of the diagram and is perpendicular to O_0O. This couple combines with $\boldsymbol{N}_0$ (not shown in the figure) to form the resultant couple $\boldsymbol{N}$ at O, whose moment is given by :

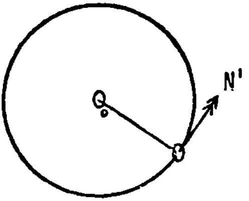

FIG. 31.

$$N^2 = N_0^2 + N'^2 = N_0^2 + O_0O^2 \cdot F^2 \,. \quad . \quad (307)$$

and whose axis lies in the plane perpendicular to O_0O and makes an angle with the plane of the diagram given by :

$$\tan\theta = \frac{N_0}{N'} \quad . \quad . \quad . \quad . \quad . \quad (308)$$

If we make the point O move round a circle described about O_0, N', N and θ remain constant. But if the distance OO_0 is increased N' and N increase to an unlimited extent, while the angle θ at the same time decreases to an indefinitely small amount. If we take point O out of the plane of the diagram into space, all the points O having a definite moment N form an infinite circular cylinder, whose radius increases to an infinite amount as N increases. The common axis of all these circular cylinders, the locus of all the points O_0, is called the *central axis* of the system of forces; for it the resultant couple N has its *smallest* value N_0.

§ 90. In addition to the most general case, we shall also consider briefly a few important special cases.

If $N_0 = 0$, the couple N corresponding to the point of application O of the force $\boldsymbol{F}$ reduces to $\boldsymbol{N}'$ (Fig. 31). Then there is no couple at all on the central axis—that is, the system of forces reduces to a force $\boldsymbol{F}$, which acts at a point on the central axis.

The condition for a system of forces to reduce to a force alone is not, then, that the resultant couple $\boldsymbol{N} = 0$ (the origin of co-ordinates O being chosen at random), but that $\boldsymbol{N}$ should be perpendicular to $\boldsymbol{F}$, or, by (306), in vectorial notation, that:

$$\Sigma \boldsymbol{F}_1 \cdot \Sigma[\boldsymbol{r}_1, \boldsymbol{F}_1] = 0 \quad . \quad . \qquad (309)$$

Actually, by choosing the point of application of the resultant appropriately, we can then always make the couple $\boldsymbol{N}$ vanish.

If, on the other hand, $\boldsymbol{F}$ vanishes, the central axis becomes indeterminate. The system of forces then reduces to a definite couple $\boldsymbol{N}$, whose point of application is arbitrary. This case is realized, for example, in the influence of the earth's magnetism on a rigid magnet.

If, lastly, $\boldsymbol{F}$ and $\boldsymbol{N}$ both vanish, we have equilibrium; here, too, the choice of the origin of co-ordinates is immaterial.

§ 91. **Bodies with Limited Freedom of Motion.** The conditions of equilibrium (306a) refer to a body which is freely movable. But if certain limits are set to the freedom of motion of the body—say by external forces of constraint—the equations (306*a*) represent sufficient but by no means necessary conditions of equilibrium, and our next question is: What are the necessary conditions in each individual case?

Let us first consider a body in which a straight line is kept fixed, and which is acted on by an arbitrary system of forces $\boldsymbol{F}_1$, $\boldsymbol{r}_1$, ... Such a body represents the most general type of lever. For let us take the fixed straight line as the z-axis, and first reduce the system of forces to the resultant $\boldsymbol{F}$ that acts at the origin of co-ordinates and the associated couple $\boldsymbol{N}$. In order that equilibrium be maintained it is not necessary that $\boldsymbol{F} = 0$; for at the origin of co-ordinates and at every point of the z-axis constraining forces act which under all circumstances neutralize the driving forces acting at these points.

Moreover, with regard to the couple $\boldsymbol{N}$, its component $\boldsymbol{N}_x$ may be represented by two anti-parallel forces which act in the direction of the y-axis at points on the z-axis, and therefore become destroyed by the external constraint. The same holds for the component $\boldsymbol{N}_y$, whose forces may be assumed to act in the direction of the x-axis at points on the z-axis. The component $\boldsymbol{N}_z$ alone cannot be annulled by the resistance of the z-axis.

Hence it is sufficient for equilibrium but at the same time necessary that:

$$\boldsymbol{N}_z = \Sigma(x_1\boldsymbol{F}_{y1} - y_1\boldsymbol{F}_{x1}) = 0 \quad . \quad . \quad . \qquad (310)$$

That is, we require only one equation between the components of the driving forces and the co-ordinates of their points of application. If $\boldsymbol{N}_z$ is not equal to zero, the driving forces bring about a motion—in this case a rotation of the body about the z-axis. For this reason the statical moment $\boldsymbol{N}_z$ of the system of forces with respect to the z-axis is also called the turning moment (*Drehungsmoment*) about this axis.

Hence whereas of the six equations (306*a*) for the equilibrium of a free rigid body only one is of use in the present instance, the other five are necessary to answer the question as to the resistance which the fixed axis must offer—that is, the constraint which must be exerted on it in order that, regarded as a freely movable body, it may remain at rest. This constraint is evidently such that it exactly cancels the action of the driving forces according to (306*a*); thus it consists of a force $-\boldsymbol{F}$ which acts at the origin of co-ordinates and a couple whose components are $-\boldsymbol{N}_x$ and $-\boldsymbol{N}_y$.

A couple which has the *z*-axis as its axis is of course unable to supply the constraint, since all the constraining forces pass through points of the *z*-axis. It is easy to see that to keep the *z*-axis fixed it is sufficient only to keep two points on it fixed, for example, the origin of co-ordinates and one other point.

Hence the forces of constraint may in this case always be reduced to two forces which act at these two points.

If the body, besides being able to turn about the *z*-axis, can also glide along this axis (we may imagine the body to be traversed by a smooth fixed rod or pin), then (310) is not sufficient for equilibrium; rather, we must add:

$$\Sigma \boldsymbol{F}_z = 0 \qquad \qquad (311)$$

For in this case the constraining force is unable to supply a component in the direction of the *z*-axis.

It is easy to see, indeed, that the more freely movable the body is, the less the constraint, the greater is the number of equations of condition which the driving forces must fulfil if equilibrium is to be maintained. This leads us to refer to the remark made in § 71 apropos of the motion of a material point. A body which can be rotated about a fixed axis has a single degree of freedom; for its position is determined by a single variable, the angle of rotation. Accordingly, a single equation of condition is sufficient to ensure equilibrium. If the body can at the same time glide along the axis of rotation, a second degree of freedom is added and, with it, a second condition for equilibrium; and we can proceed further in this way, as will be seen in the next chapter.

Suppose the body can be rotated about a fixed point and let us take this point as the origin of co-ordinates O. Then the resultant $\boldsymbol{F}$ that acts at O will be cancelled by the constraint, and the necessary and sufficient condition for equilibrium is:

$$\boldsymbol{N} = \Sigma[\boldsymbol{r}_1, \boldsymbol{F}_1] = 0 \quad . \quad . \quad . \quad . \qquad (312)$$

This represents three equations between the components of the driving forces and the co-ordinates of their points of application.

We shall see that such a body also has three degrees of freedom. If $\boldsymbol{N}$ differs from zero, the driving forces effect a rotation of the body about O. Hence the statical moment $\boldsymbol{N}$ of a system of forces with respect to a point O is also called the "moment of rotation" of the forces about this point.

CHAPTER II

STATICS OF ANY ARBITRARY SYSTEM OF POINTS

§ 92. We shall now generalize the laws of statics of a rigid body for the case of an arbitrary system of material points (or point-masses). For this purpose we first propose the problem of finding the conditions for the equilibrium of a system of n point-masses, on which given driving forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, . . . $\boldsymbol{F}_n$ act and whose motions are subject from the outset to certain restrictions. We suppose these conditions to be represented by a certain number p of equations between the co-ordinates of the points. Our problem then includes as special cases the statics of a single material point, considered in Part One, and the statics of a rigid body, since a rigid body is nothing else than a system of points whose distances are kept constant.

The system here assumed has $3n-p$ degrees of freedom. For of the total $3n$ co-ordinates only $3n-p$ are freely variable; the remaining p co-ordinates are determined by the prescribed conditions. The number p cannot be greater than the number $3n$. In the limiting case, $p = 3n$, *all* the points are fixed, since their positions are already determined by the conditions; in the opposite limiting case, $p = 0$, all the points are free.

To solve the proposed problem we follow the same line of reasoning as led us to a successful conclusion in the case of a single material point. We take into account the physical influence of the prescribed conditions by introducing constraining forces $\boldsymbol{Z}$, which represent this influence; for in no other way but by forces can this influence manifest itself.

After having introduced these constraining forces we may regard the points as free, and we obtain as the condition of equilibrium for the system of points the following $3n$ equations between the force-components :

$$\boldsymbol{F}_1 + \boldsymbol{Z}_1 = 0, \quad \ldots \quad (313)$$

Here $\boldsymbol{Z}_1$ denotes the resultant of the constraining forces which are called up at the point 1 by all the p conditions. If in particular one of the equations of condition does not contain the co-ordinates of the point 1, it of course makes no contribution to $\boldsymbol{Z}_1$.

The form (313) of the equation of condition remains unfruitful so long as we know nothing further of the constraining forces that have been introduced. We therefore next endeavour to set up as general as possible a property of the constraining forces. In the mechanics of a single material point with restricted freedom of motion we found that the constraining force always acts perpendicularly to the fixed curve or the fixed surface, and that the work done by the constraining force during any motion of the point that may occur always vanishes. The first form of this theorem is not capable of being generalized to apply to the system of points here in question, since the prescribed conditions may be quite different from those for fixed curves or surfaces. But it is possible to deduce quite generally the theorem that in any motion of the system of points under the influence of arbitrary driving forces the work of all the forces of constraint at all points taken together must always be equal to zero, or :

$$\Sigma \boldsymbol{Z}_1 \cdot d\boldsymbol{r}_1 = 0 \quad \ldots \quad (314)$$

where $d\boldsymbol{r}_1$, as in (149), denotes the vectorial distance traversed in the time dt by the point 1.

§ 93. The equation (314) forms the basis of the whole statics of non-free point-systems. To prove them we must investigate more closely the physical meaning of the p equations of condition for the co-ordinates of the points. We can do this only by supposing these equations to be realized physically in some way. Such considerations can by no means be circumvented, for the equations in themselves cannot exert constraining forces at all. They have a physical meaning only if they are regarded as the comprehensive expression for the mode of action of certain real mechanisms.

We shall first give the proof of equation (314) for a few simple cases. For a single point ($n = 1$, $p = 0, 1, 2, 3$) the equation has been fully shown to be valid in Chapter VI of Part One (§ 67).

Let us now take two point-masses and first consider the special case where both points are connected by a rigid mass-less straight line (rod) of length l, but are otherwise free. Then the following equation of condition holds between the co-ordinates :

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = l^2 \quad . \qquad (315)$$

What do we know in this case, for any arbitrary motion of the two points under the influence of arbitrary driving forces, of the forces of constraint $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ which by virtue of the rigidity of the connecting straight line act at the two points 1 and 2 ?

If $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are introduced as special forces, we may, without the motion being affected, regard the two points as free. But of course we need not do this; thus if the points remain rigidly connected they move under the influence of the driving forces exactly in the same way whether the constraining forces $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are specially introduced or not. Hence these two forces acting on the rigid system cancel each other out; this requires that they shall be equal and opposite and that their directions shall be the same as that of the line connecting the two points. Thus :

$$\left.\begin{aligned} \boldsymbol{Z}_{x_1} &= S \cdot \frac{x_2 - x_1}{l}, \quad \boldsymbol{Z}_{x_2} = - S \cdot \frac{x_2 - x_1}{l} \\ \boldsymbol{Z}_{y_1} &= S \cdot \frac{y_2 - y_1}{l}, \quad \boldsymbol{Z}_{y_2} = - S \cdot \frac{y_2 - y_1}{l} \\ \boldsymbol{Z}_{z_1} &= S \cdot \frac{z_2 - z_1}{l}, \quad \boldsymbol{Z}_{z_2} = - S \cdot \frac{z_2 - z_1}{l} \end{aligned}\right\} \qquad (316)$$

where S, the tension of the rigid rod, denotes the value of the constraining force, being positive when a pull is being exerted, and negative when the force is a pressure.

By (316) the total work of the constraining forces is :

$$\begin{aligned}\boldsymbol{Z}_{x_1}dx_1 + \boldsymbol{Z}_{y_1}dy_1 + \boldsymbol{Z}_{z_1}dz_1 + \boldsymbol{Z}_{x_2}dx_2 + \boldsymbol{Z}_{y_2}dy_2 + \boldsymbol{Z}_{z_2}dz_2 \\ = -\frac{S}{l}\Big\{(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) \\ + (z_2 - z_1)(dz_2 - dz_1)\Big\} \quad (317)\end{aligned}$$

and this expression vanishes for all times, as we see immediately if we differentiate (315) with respect to the time.

We next assume the two rigidly connected points no longer to be free, but to be further restricted in their motion by being compelled to remain on a fixed curve.

The equation (314) then continues to remain valid. For the sum-total of all the constraining forces is the sum of the amounts of work done by the individual forces of constraint, and as such we here have, besides the tension of the rigid straight line, only the resistances of the fixed curves, for which the theorem has already been proved.

In the same way we can dispose of the more general case of a series of point-masses, each of which moves on a fixed curve and, besides, is connected with the preceding and the following point by means of a rigid mass-less straight line (except the first and last point, which are not connected with each other). In this case, too, the total work of all the constraining forces is zero in any motion that may occur.

The system of points just considered has a single degree of freedom. For the motion of the first point on its curve, which depends on a single variable, completely determines the motion of all the other points, as we see immediately if, starting out from the first point, we look for the position of the second, third, etc., points and reflect that each succeeding point, besides lying on its own curve, also lies on a spherical surface which is described about the preceding point with a definite radius —namely, the length of the connecting line.

§ 94. After these preliminaries we shall now give a systematic proof of equation (314)—first for a point-system having a single degree of freedom.

The case $n = 1$ has already been considered (a point-mass on a fixed curve). The case $n = 2$ corresponds to two point-masses and five equations of condition between the six co-ordinates.

In order that the equations of condition may have a physical sense we must realize them in some way by means of a mechanism. This may be done as follows. If we eliminate from the five equations of condition the co-ordinates of the second point we obtain for the co-ordinates of the first point two equations, which represent a fixed curve, on which the point is constrained to remain. We imagine this curve to be realized as a material thing and that the point 1 is kept attached to it (cf. § 65). In the same way, we realize the fixed curve calculated from the equations of condition for the point 2. All that remains for us to do is to find an appropriate mechanism to compel the point 2 to move on its curve in a perfectly definite way prescribed by the equations of condition, when the point 1 moves in any known way. If we were to connect the points rigidly together, this condition would be much too special; so it would not be able to fulfil its purpose. The following construction, however, always leads to the desired goal. At each of the point-masses 1 and 2 we attach a mass-less straight line of arbitrary lengths l_1 and l_2, which must not, however, be too small, and we connect together the other end-points of these two straight lines in such a way that they can move freely against each other at their meeting-point P. If, now, the point-masses 1 and 2 move in a manner which satisfies the five equations of condition, the point P is still able to move in very different ways. From these curves we now choose a definite one and regard it as being materialized, so that we may compel the point P to remain on it. The three points 1, P, 2 then form a mechanical system with a single degree of freedom after the manner of that considered in the preceding section, whose single point-masses 1 and 2 are subject to the five prescribed equations of condition. Thus this mechanical system represents a realization in material form of the five given equations of condition and is fully equivalent to them physically. If we refused to recognize this conclusion we should not be able to attribute a definite physical meaning to the equations of condition at all.

In the preceding section we proved the validity of equation (314) for the forces of constraint in a system of points of the kind in question; consequently it also holds quite generally for the forces of constraint, conditioned by the five equations, which act at the points 1 and 2. For, as concerns the specially introduced point P, it is true that forces of constraint also act on it which are due to its fixed rigid curve and to the rigid straight lines l_1 and l_2. But their resultant $\boldsymbol{Z}$ and hence also the work done by them is, by equation (217), equal to zero for any arbitrary motion of the point P, because the point P has the mass $m = 0$ and because the driving force $\boldsymbol{F}$ which acts on it is also zero.

If we have the case $n = 3$—that is, three point-masses with eight equations of condition between their co-ordinates—these equations may also be realized by a mechanism of the kind described, by making the point-masses 1, 2, 3 move on the fixed curves and by interposing a point P between 1 and 2 and also another between 2 and 3, just as before. Then the same considerations lead to the same goal; and the case of any arbitrary number n of point-masses with one degree of freedom may be disposed of in the same way.

We have still to consider a point-system with several degrees of freedom. If such a system performs any motion under the action of any driving forces, clearly we can, if we assume the motion to be known, add to the existing p equations of condition between the co-ordinates any further arbitrary number of arbitrary equations as new fixed conditions between the co-ordinates, which are compatible with the motion, without the motion being disturbed. But these additional conditions play only a formal part, since they are actually superfluous.

If we now suppose $3n - p - 1$ such new conditions to be introduced, the total number of conditions amounts to $3n - 1$, and the point-system has a single degree of freedom; thus the equation (314) is fulfilled for the total work done by the forces of constraint arising from all the real conditions. But since the forces of constraint of all the new conditions that have been introduced are equal to zero, the expression for the total work reduces to the work done by the forces of constraint which arise from the real conditions, and so the equation (314) is proved to be valid quite generally. We can express it in words as follows: *forces of constraint may individually do work or use up work but never when taken as a whole.* This law is intimately connected with the principle of conservation of energy; for so long as the continual maintenance of the fixed conditions which we realize mechanically entails no gain or loss of work, no gain or loss of work can arise from their method of acting (cf. § 75 in this connection).

§ 95. On the basis of (314) we may now make exactly the same deductions for an arbitrary system of points as in § 67 for an individual point. But since the line of reasoning is exactly the same in both cases, we need here only state the results. If a system of point-masses which was originally at rest, and whose co-ordinates are restricted by several prescribed conditions, is set into motion by the driving forces that act on it, each point-mass moves in accordance with § 76 in the direction of the resultant of the driving forces $\boldsymbol{F}$ that act on it and the forces of constraint $\boldsymbol{Z}$. Hence we have for the infinitesimal displacements of the points that occur in the first element of time:

$$\Sigma(\boldsymbol{F}_1 + \boldsymbol{Z}_1) \,.\, d\boldsymbol{r}_1 > 0 \,. \quad . \quad . \quad . \qquad (318)$$

and in view of (314):

$$\Sigma \boldsymbol{F}_1 . d\boldsymbol{r}_1 > 0 \quad . \quad . \quad . \quad . \quad . \qquad (319)$$

That is, *when motion begins to occur the driving forces on the whole perform positive work.* Thus motion can occur in a system of points at rest only if the points can perform a displacement for which the work of the driving forces is positive. If the prescribed conditions are such that the points are unable to undergo a displacement at all in which the driving forces perform positive work, no motion can occur at all and the whole system persists at rest—that is, in equilibrium. Hence we get as a sufficient condition for the equilibrium of the point-system that for *every* infinitesimal displacement of the co-ordinates which is compatible with the given equations:

$$\Sigma \boldsymbol{F}_1 . \delta \boldsymbol{r}_1 \leqq 0 \quad . \quad . \quad . \quad . \quad . \qquad (320)$$

Here $\delta \boldsymbol{r}$ denotes a *perfectly arbitrary* displacement among all those compatible with the prescribed conditions, and is therefore called a *virtual* displacement, in contrast with the *real* displacement $d\boldsymbol{r}$ which occurs in the element of time dt. So the equation (320) is called the *Principle of Virtual Displacements or the Principle of Virtual Work.* It was discovered by John Bernoulli in 1717.

In the cases here under investigation the expression for the principle may be considerably simplified. For since the prescribed conditions are expressed by equations (and not by inequalities between the co-ordinates of the points), we see that for every possible system of virtual displacements $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$, . . . the exactly opposite system of displacements $- \delta \boldsymbol{r}_1$, $- \delta \boldsymbol{r}_2$, . . . is possible. Now if the points are in a position for which a system of virtual displacements can be effected which allow negative work to be performed, there is certainly also a system of virtual displacements for which positive work may be performed—namely, exactly the opposite; and so it is possible for motion to occur in the direction in question. Hence the equilibrium is guaranteed for all directions only if for *every* system of virtual displacements :

$$\Sigma \boldsymbol{F}_1 . \delta \boldsymbol{r}_1 = \Sigma \boldsymbol{F}_{x_1} \delta x_1 + \boldsymbol{F}_{y_1} \delta y_1 + \boldsymbol{F}_{z_1} \delta_{z_1} = 0 . \qquad (321)$$

§ 96. The significance of the principle of virtual work consists essentially in the circumstance that in order to find the conditions of equilibrium we need not enter in any way either into the mechanisms by which the prescribed conditions are realized nor into the constraining forces which arise from them. It is quite sufficient to know all the kinds of displacement which the given conditions allow to the points which are subject to them. Moreover, the principle has the important practical advantage that it combines the entire set of conditions of equilibrium in a single equation—a result which is attained only because this is no ordinary equation, but a variational equation which holds not for definite but for any arbitrary quantities. For it is clear that the content of such a variational equation is so much the richer and the conditions which it postulates are so much the more comprehensive the more arbitrarily we may choose the variations which must obey it.

For example, if the variations $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$, . . . may all be chosen quite arbitrarily—that is, if all the points are free—(321) can be fulfilled only if :

$$\boldsymbol{F}_1 = 0, \quad \boldsymbol{F}_2 = 0, \quad . \ . \ .$$

for there is nothing to prevent our taking all the variations of all the co-ordinates equal to zero except for one single variation, say δx_1. Of the virtual work only the one term $\boldsymbol{F}_{x_1} . \delta x_1$ then remains, and since the virtual work is to be equal to zero, the first factor $\boldsymbol{F}_{x_1}$ must vanish in the product mentioned. In this way the principle of virtual work leads us to the well-known conditions of equilibrium for a system of free points.

The opposite extreme is that where all the points are fixed—that is, their co-ordinates are already given by the prescribed conditions. Then the permissible displacements δr are all equal to zero, and the condition for equilibrium (321) is identically fulfilled for every arbitrary value of the driving forces, so that equilibrium exists in all circumstances, as direct evidence demands.

In general, for any arbitrary number p of prescribed conditions between the $3n$ point co-ordinates—that is, in the case of a system of $3n - p$ degrees of freedom, we arrive from the variational equation (321) at the finite conditions of equilibrium between the force components and the point components by first reducing the $3n$ variations $\delta x_1, \delta y_1, \delta x_2, \ldots$ by means of the p given equations of condition which we shall denote by $f = 0$, $\phi = 0$, $\psi = 0, \ldots$ to $3n - p$ arbitrarily selected variations, which are then quite independent of one another. This is done by solving the p homogeneous linear equations of condition :

$$\left.\begin{array}{l}
\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial y_1}\delta y_1 + \frac{\partial f}{\partial z_1}\delta z_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \frac{\partial f}{\partial y_2}\delta y_2 + \ldots = 0 \\
\frac{\partial \phi}{\partial x_1}\delta x_1 + \frac{\partial \phi}{\partial y_1}\delta y_1 + \frac{\partial \phi}{\partial z_1}\delta z_1 + \frac{\partial \phi}{\partial x_2}\delta x_2 + \frac{\partial \phi}{\partial y_2}\delta y_2 + \ldots = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad = 0
\end{array}\right\} \quad (322)$$

in terms of the p variations, which are to be regarded as dependent on the remaining $3n - p$ variations.

By substituting these values in (321) we then obtain the virtual work as a homogeneous linear function of the $3n - p$ variations which are independent of one another, and according to the above reflection concerning a system of independent variations the vanishing of the virtual work demands that each individual coefficient of each variation that is independent of the other variations must be zero.

In this way we obtain just as many equations of condition between the force-components and the point-co-ordinates as there are independent variations—that is, degrees of freedom, namely $3n - p$—and so we have generalized the theorem which we have already found to hold for a single material point (§ 71) as well as for a rigid body (§ 91).

§ 97. If we endeavour to carry out the calculation in the manner just described, we find in general that very awkward operations are involved. We have, however, in Lagrange's method of elimination (by means of undetermined multipliers) a method of great value for arriving at the result in a way which, although indirect, can easily be followed.

We multiply equations of condition (322), after the operation of variation, successively by certain quantities λ, μ, ν, . . ., the choice of which we leave open, and then add them to equation (321). We then get as the equation for equilibrium :

$$\Sigma\left(\boldsymbol{F}_{x_1} + \lambda\frac{\partial f}{\partial x_1} + \mu\frac{\partial \phi}{\partial x_1} + \nu\frac{\partial \psi}{\partial x_1} + \ . \ . \ . \ \right)\delta x_1 = 0 \qquad (323)$$

where the summation is to be performed over all $3n$ co-ordinates; this equation holds for all arbitrary virtual displacements and for all arbitrary values of the p quantities λ, μ, ν . . . .

We next choose these p quantities so that the bracketed coefficients of the first p variations, starting from δx_1, vanish.

The virtual work (323) then reduces to a linear homogeneous function of the $3n - p$ remaining variations, and since we may regard these as completely independent of one another, the variational equation, just as above, demands that the coefficients of these $3n - p$ variations vanish individually.

The net result is simply that all $3n$ coefficients of the expression (323) may be set equal to zero :

$$\left.\begin{array}{l} \boldsymbol{F}_{x_1} + \lambda\dfrac{\partial f}{\partial x_1} + \mu\dfrac{\partial \phi}{\partial x_1} + \nu\dfrac{\partial \psi}{\partial x_1} + \ . \ . \ . = 0 \\ \boldsymbol{F}_{y_1} + \lambda\dfrac{\partial f}{\partial y_1} + \mu\dfrac{\partial \phi}{\partial y_1} + \nu\dfrac{\partial \psi}{\partial y_1} + \ . \ . \ . = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right\} \qquad (324)$$

for all points and co-ordinates.

When λ, μ, ν . . . have been eliminated, these equations are actually the equations of condition between the force-components and the point-co-ordinates, and so represent the desired conditions of equilibrium in a symmetrical and concise form.

§ 98. By comparing the conditions of equilibrium (324) with the conditions of equilibrium (313), we may arrive directly at the values of the constraining forces—for example, of the x-component of the resultant of all the constraining forces that act on the point 1 :

$$Z_{x_1} = \lambda \frac{\partial f}{\partial x_1} + \mu \frac{\partial \phi}{\partial x_1} + \nu \frac{\partial \psi}{\partial x_1} + \ldots \quad (325)$$

The individual terms refer to the forces of constraint that originate in the individual conditions. If a co-ordinate of a point does not occur at all in an equation of condition, this condition furnishes no corresponding component for the constraining force that acts on the point.

If, on the other hand, we inquire into the various forces of constraint which are exerted owing to a definite condition—for example, $f = 0$—on the different points, their components are in the ratio :

$$\frac{\partial f}{\partial x_1} : \frac{\partial f}{\partial y_1} : \frac{\partial f}{\partial z_1} : \frac{\partial f}{\partial x_2} : \frac{\partial f}{\partial y_2} : \ldots \quad . \quad (326)$$

which represents a generalization of the expression (246) which is valid for a single material point on a fixed surface.

§ 99. We shall now apply the principle of virtual work to the equilibrium of a free or non-free rigid body. Although we have already treated this case above, it is interesting to apply a new method to a problem which has been solved elsewhere, because in this way we become aware of its peculiar properties, and are often led to answer a series of new questions, which is of use for treating other problems later.

A rigid body is a system of material points whose mutual distances remain constant: it is immaterial here whether the points are finite in number or whether they fill space continuously as infinitesimal elements of mass.

We next inquire into the condition for the equilibrium of a rigid body which is able to rotate about a *fixed axis* and on which definite driving forces act at definite points of application.

If we wished to write down all the equations of condition $f = 0$, $\phi = 0$, . . . and to apply the above described method of Lagrange, we should have to deal with very long calculations. It is much more convenient to apply the method first described in § 96 and to refer the variations of all the point-co-ordinates directly to just as many independent variations as there are degrees of freedom.

Now, a rigid body with a fixed axis clearly has only a single degree of freedom. For its position is determined when we know the angle which a fixed plane described through the axis of rotation and lying in the body makes with a plane which likewise contains the axis of rotation and is at rest in space.

Hence all the virtual displacements must be expressible by a single variation—namely, by the infinitely small angle of rotation.

This is achieved most simply by introducing cylindrical co-ordinates: ρ, ϕ, z, where, as before, we take the axis of rotation as the z-axis.

Then we have for a point of the rigid body:

$$x_1 = \rho_1 \cos\phi_1, \quad y_1 = \rho_1 \sin\phi_1, \quad z_1 = z_1$$

and for the variations of the co-ordinates, since ρ_1 and z_1 remain constant when the body is rotated about the z-axis:

$$\left.\begin{aligned} \delta x_1 &= -\rho_1 \sin\phi_1 \delta\phi_1 \\ \delta y_1 &= \rho_1 \cos\phi_1 \delta\phi_1 \\ \delta z_1 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (326a)$$

Now $\delta\phi_1$ has exactly the same value for all points of the body—namely, the value of the angle of rotation, which we shall denote by ζ; hence:

$$\delta x_1 = -y_1 \cdot \zeta, \quad \delta y_1 = x_1 \cdot \zeta, \quad \delta z_1 = 0 \quad . \quad (326b)$$

and if we substitute these values in (321) we get the following expression for the work performed by the driving forces during an infinitesimal rotation of the body through the angle ζ:

$$\zeta \cdot \Sigma(x_1 \boldsymbol{F}_{y_1} - y_1 \boldsymbol{F}_{x_1}) = \zeta \cdot \boldsymbol{N}_z \quad . \quad . \quad . \quad (327)$$

According to § 91, this is the product of the angle of rotation and the angular momentum of the driving forces about the z-axis. If the driving forces set the originally stationary body into rotation, then by (319) the work done by the driving forces is positive—that is, the rotation occurs in the sense of the angular momentum.

Hence if the angular momentum is zero, the driving forces are unable to do any work at all, and the body must remain at rest. We therefore again obtain as the condition of equilibrium the equation (310), but in a much simpler manner, formally, than before.

If the body can also *glide* along the axis of rotation (cf. § 91), then its displacement is more general, being dependent on two variations—namely, that of the angle of rotation ζ and that of the distance glided w, which is common to all points of the body. The variations of the point-co-ordinates then become:

$$\delta x_1 = -y_1 \cdot \zeta, \delta y_1 = x_1 \cdot \zeta, \delta z_1 = w$$

and the principle of virtual work (321) gives:

$$\zeta \cdot \Sigma(x_1 \boldsymbol{F}_{y_1} - y_1 \boldsymbol{F}_{x_1}) + w \cdot \Sigma \boldsymbol{F}_{z_1} = 0 \quad . \quad . \quad (328)$$

which, since ζ and w are independent of each other, give the two conditions of equilibrium (310) and (311).

§ 100. We now assume that only *one* point is *fixed* in the rigid body and that the body can be freely rotated about this point. Our first question is: what is the number of degrees of freedom of this system? To characterize the position of the body it is not sufficient to specify the position of one of its movable points. Hence, as in § 56, we introduce a second right-handed rectilinear co-ordinate system x', y', z' which is fixed in the body and so moves with it. We make the origin coincide with the origin of the stationary co-ordinate system x, y, z, which is situated at the fixed point. The position of the body is then determined by the position of the "accented" system and depends only on the nine direction-cosines $\alpha_1, \ldots \gamma_3$ (§ 56).

In the equations of transformation :

$$\left.\begin{aligned} x &= \alpha_1 x' + \alpha_2 y' + \alpha_3 z' \\ y &= \beta_1 x' + \beta_2 y' + \beta_3 z' \\ z &= \gamma_1 x' + \gamma_2 y' + \gamma_3 z' \end{aligned}\right\} \quad . \qquad (329)$$

the accented co-ordinates are independent of the position of the body for a definite material point of the body, and hence the variations of the unaccented co-ordinates are :

$$\left.\begin{aligned} \delta x &= x'\delta\alpha_1 + y'\delta\alpha_2 + z'\delta\alpha_3 \\ \delta y &= x'\delta\beta_1 + y'\delta\beta_2 + z'\delta\beta_3 \\ \delta z &= x'\delta\gamma_1 + y'\delta\gamma_2 + z'\delta\gamma_3 \end{aligned}\right\} \quad . \quad . \qquad (330)$$

But the variations of the direction-cosines do not yet represent the independent variations. For, like the direction-cosines themselves, they are connected together among themselves by a series of relationships.

By (32) we have :

$$\left.\begin{aligned} \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1 \\ \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1 \end{aligned}\right\} . \quad . \quad . \qquad (331)$$

and, besides, since the accented axes form a right-angled system, by (37) :

$$\left.\begin{aligned} \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 &= 0 \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0 \\ \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \qquad (332)$$

Altogether we thus have six relationships, from which we gather that of the nine direction-cosines only three may be chosen arbitrarily and then the other six are determined by them. Thus the movable body has three degrees of freedom, and accordingly we must expect three conditions of equilibrium. To find them we must refer the co-ordinate-variations (330) to three independent variations common to all points of the body.

It would be inexpedient to select from the nine variations $\delta\alpha_1$, . . . any three arbitrarily as independent variations, because this would destroy the symmetry of the equations. We do better to proceed indirectly by first replacing the accented co-ordinates in (330) again by unaccented co-ordinates, in accordance with the equations (181). We then get:

$$\begin{aligned}
\delta x &= (\alpha_1\delta\alpha_1 + \alpha_2\delta\alpha_2 + \alpha_3\delta\alpha_3)x + (\beta_1\delta\alpha_1 + \beta_2\delta\alpha_2 + \beta_3\delta\alpha_3)y \\
&\qquad + (\gamma_1\delta\alpha_1 + \gamma_2\delta\alpha_2 + \gamma_3\delta\alpha_3)z \\
\delta y &= (\alpha_1\delta\beta_1 + \alpha_2\delta\beta_2 + \alpha_3\delta\beta_3)x + (\beta_1\delta\beta_1 + \beta_2\delta\beta_2 + \beta_3\delta\beta_3)y \\
&\qquad + (\gamma_1\delta\beta_1 + \gamma_2\delta\beta_2 + \gamma_3\delta\beta_3)z \\
\delta z &= (\alpha_1\delta\gamma_1 + \alpha_2\delta\gamma_2 + \alpha_3\delta\gamma_3)x + (\beta_1\delta\gamma_1 + \beta_2\delta\gamma_2 + \beta_3\delta\gamma_3)y \\
&\qquad + (\gamma_1\delta\gamma_1 + \gamma_2\delta\gamma_2 + \gamma_3\delta\gamma_3)z
\end{aligned}$$

On account of the relationships that exist between the direction-cosines and their variations we may now effect considerable simplifications.

In the first place, it is easy to see from geometrical considerations that the relationships (331) and (332) retain their validity if we exchange in them the figures 1, 2, 3 by the letters α, β, γ—that is, the accented co-ordinate-axes by the unaccented axes. The analogous relationships then follow:

$$\left.\begin{aligned}
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 &= 1 \\
\beta_1^2 + \beta_2^2 + \beta_3^2 &= 1 \\
\gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1
\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (333)$$

and :

$$\left.\begin{aligned}\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3&=0\\ \beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3&=0\\ \gamma_1\alpha_1+\gamma_2\alpha_2+\gamma_3\alpha_3&=0\end{aligned}\right\} \quad . \qquad (334)$$

which of course lead to nothing essentially new, being already contained in (331) and (332). By performing variation on these equations we get :

$$\left.\begin{aligned}\alpha_1\delta\alpha_1+\alpha_2\delta\alpha_2+\alpha_3\delta\alpha_3&=0\\ \beta_1\delta\beta_1+\beta_2\delta\beta_2+\beta_3\delta\beta_3&=0\\ \gamma_1\delta\gamma_1+\gamma_2\delta\gamma_2+\gamma_3\delta\gamma_3&=0\end{aligned}\right\} \quad . \quad . \qquad (335)$$

and :

$$\left.\begin{aligned}\alpha_1\delta\beta_1+\alpha_2\delta\beta_2+\alpha_3\delta\beta_3&=-(\beta_1\delta\alpha_1+\beta_2\delta\alpha_2+\beta_3\delta\alpha_3)=\zeta\\ \beta_1\delta\gamma_1+\beta_2\delta\gamma_2+\beta_3\delta\gamma_3&=-(\gamma_1\delta\beta_1+\gamma_2\delta\beta_2+\gamma_3\delta\beta_3)=\xi\\ \gamma_1\delta\alpha_1+\gamma_2\delta\alpha_2+\gamma_3\delta\alpha_3&=-(\alpha_1\delta\gamma_1+\alpha_2\delta\gamma_2+\alpha_3\delta\gamma_3)=\eta\end{aligned}\right\} . \qquad (336)$$

if we introduce for the sake of brevity the infinitesimal quantities ξ. η, ζ, which correspond in their notation to the letters α, β, γ or the unaccented co-ordinates x, y, z, respectively.

The above equations for the co-ordinate-variations then run singly :

$$\left.\begin{aligned}\delta x&=\eta z-\zeta y\\ \delta y&=\zeta x-\xi z\\ \delta z&=\xi y-\eta x\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (337)$$

and in this form they appear simply reduced to terms of three independent variational quantities ξ, η, ζ common to all points of the body.

Substituting in (321) we get for the virtual work (321) of the driving forces, if we use the abbreviation (306) :

$$\xi\cdot\boldsymbol{N}_x+\eta\cdot\boldsymbol{N}_y+\zeta\cdot\boldsymbol{N}_z, \quad . \quad . \quad . \qquad (338)$$

and the condition for equilibrium is the vanishing of all three components of $\boldsymbol{N}$, as in (312).

§ 101. The equations (337) for the most general displacement of the points of a rigid body with a fixed origin of co-ordinates may, by (303), be written in vectorial form as:

$$\delta \boldsymbol{r} = [\boldsymbol{o}, \boldsymbol{r}] \quad . \quad . \quad . \quad . \quad . \qquad (339)$$

if we take $\boldsymbol{o}$ as standing for the vector whose components are the infinitesimal quantities ξ, η, ζ. The simplicity of this formula suggests that the vector $\boldsymbol{o}$ has an important kinematical meaning; we shall now investigate this more closely.

For the special case $\xi = 0$, $\eta = 0$ the equations (337) become:

$$\delta x = -\zeta y,\ \delta y = \zeta x,\ \delta z = 0 \quad . \quad . \quad . \qquad (340)$$

These are precisely the expressions (326*b*) for the co-ordinate-variations when a rigid body is rotated about the z-axis through the infinitesimal angle ζ. In the same way the equations:

$$\delta x = 0,\ \delta y = -\xi z,\ \delta z = \xi y, \quad . \quad . \quad . \qquad (341)$$

$$\delta x = \eta z,\ \delta y = 0,\ \delta z = -\eta x \quad . \quad . \quad . \qquad (342)$$

represent rotations of the body about the x- and the y-axis, through the infinitesimal angles ξ and η.

It is easy to see that we obtain the most general displacement (337), if we add together the variations (340), (341) and (342) for each individual co-ordinate—that is, if we subject the body successively to the three rotations mentioned, but in any order of sequence. We must take care to note, however, that when the first rotation has been carried out, say about the z-axis, the material point which was originally situated at the point x, y, z now has the co-ordinates $x - \zeta y$, $y + \zeta x$, z, and that therefore *these* values and *not* the values x, y, z are to be substituted in the equations (341) for the second rotation, if we wish to find out at what point of space the material point which was originally situated at x, y, z finally arrives as a result of the three successive rotations. But we may convince ourselves by performing the calculations directly that the error caused by neglecting this circumstance is one of a higher order of infinitesimals. For the more exact equations run, for the first two rotations taken together :

$$\delta x = -\zeta y, \ \delta y = \zeta x - \xi z, \ \delta z = \xi(y + \zeta x).$$

Here the term in $\xi\zeta$ is an infinitesimal of the second order and may therefore be neglected. The same holds for the third process of rotation.

Nevertheless, we see from these considerations that in the case of finite angles of rotation the result of the successive rotations would no longer be independent of the order of sequence in which the rotations are carried out. This is a special case of the general law which allows us to superpose small events without their disturbing one another, which resolves itself ultimately into the mathematical theorem that a function of several variables, so long as they are infinitely small, is a linear function.

If we now fix our attention on the final position assumed by the body after the three rotations ξ, η, ζ have been performed, we get a very simple view of it if we calculate the displacement which a material point, originally situated on the straight line :

$$x : y : z = \xi : \eta : \zeta \quad . \quad . \quad . \quad . \quad (343)$$

has undergone as a whole. This straight line passes through the origin of co-ordinates and in general makes finite direction-angles with the co-ordinate axes. For it the equations (337) give :

$$\delta x = 0, \ \delta y = 0, \ \delta z = 0$$

That is, the point is situated in the same place at the end as at the beginning. Hence the total displacement ξ, η, ζ simply represents a rotation of the body about the straight line (343), and we get the theorem that the *most general infinitesimal displacement of a rigid body with one fixed point is a rotation about a straight line which passes through this point.* Of course, both the direction of the straight line and the magnitude of the associated angle of rotation are fully determined by the quantities ξ, η,

ζ, and by (343) the direction of the straight line is the direction of the vector $\boldsymbol{o}$. The magnitude of the angle of rotation results if we take the magnitude of the displacement:

$$|\delta \boldsymbol{r}| = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$$

from (337) and divide it by the distance of the point x, y, z from the axis of rotation:

$$\boldsymbol{r} \, . \sin (\boldsymbol{r}, \boldsymbol{o}).$$

Calculation then gives, exactly as in (304) and (305), the following value for the angle of rotation:

$$\sqrt{\xi^2 + \eta^2 + \zeta^2},$$

that is, the absolute value $|\boldsymbol{o}|$ of the vector $\boldsymbol{o}$. We therefore also say: "the three rotations ξ, η, ζ compound in magnitude and direction into a single resultant rotation in accordance with the law of the parallelogram of forces." This of course has only the sense that the rotations, when performed in any arbitrary order of succession, lead to a final position of the body which may also be arrived at by a single rotation with the characteristics above noted. For in all these reflections there is no question of force effects.

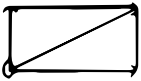

FIG. 32.

According to these theorems, we may completely symbolize an infinitesimal rotation and the composition of several such rotations graphically by means of a directed straight line which starts out from the fixed point O, its length giving the magnitude of the angle of rotation, on a convenient scale, and its direction being designated by the axis of rotation in the sense defined in § 83. To avoid confusion with the symbol for a force we may indicate the end-point of the straight line by means of a rounded arrow instead of a sharp arrow (Fig. 32). We may now operate with these symbols exactly as with those for forces; in particular, several rotations about a common axis may also be compounded simply by adding up algebraically the angles of rotation to a resultant rotation, as we see immediately if we imagine the rotations to be executed in succession.

From this we also obtain the answer to the question as to the final position assumed by the body after it has been subjected to an arbitrary number of given (infinitesimal) rotations in succession. If $\boldsymbol{o}_1, \boldsymbol{o}_2, \boldsymbol{o}_3, \ldots$ are the individual given rotations, the final result is a single rotation :

$$\boldsymbol{o} = \Sigma \boldsymbol{o}_1 \tag{344}$$

exactly as in equation (67).

The condition that the final position of the body should coincide with its initial position or that all rotations should mutually cancel is : $\boldsymbol{o} = 0$.

§ 102. All the theorems derived in the preceding section refer to a rigid body with one fixed point. Hence the axes of all the rotations hitherto considered pass through this point. The next question that suggests itself is : how do we compound infinitesimal rotations whose axes do not intersect ? To answer this question we must of course from now on imagine the rigid body to be completely *free*.

The peculiar advantage of the method used in the preceding section for compounding forces now manifests itself. For although we are not concerned now with forces at all, but with displacements, the problem here to be treated, taken formally, amounts to exactly the same as the former problem ; this is shown in the complete agreement both of the starting points and of the auxiliary methods used in the solution.

In the first place, it is clear that the symbol of a rotation (Fig. 32) may be displaced arbitrarily in its own direction without its kinematic meaning being altered, so long as the initial point of the straight line lies on the axis of rotation. This corresponds precisely with the displacement of the point of application of a force in the direction of the force. On the other hand, the initial point of the straight line cannot be displaced laterally; for rotations about parallel axes are no more identical than are parallel forces.

If we next reflect that for the whole development of the theory of forces acting on a rigid body we used no other foundations, in §§ 78 to 90, than those which are also used for infinitesimal rotations, it immediately becomes clear that for rotations we here get the same result by the same method as we there obtained for forces, and that consequently it is quite sufficient to state the results at once and to refer to the earlier discussion for details. Hence we may immediately enunciate the following theorems, all of which of course refer only to infinitesimal rotations.

Rotations about parallel axes, if they are in the same sense, compound by addition of the angles of rotation into a single rotation about an axis parallel to each. But if the rotations are in opposite senses (anti-parallel) a single rotation also results in general when the angles of rotation are added together algebraically. There is an exception, however, in the case where the algebraic sum of the angles of rotation is zero. Then the rotations either all cancel out or, more generally, there are left two equal anti-parallel rotations, which are called a "rotation-couple" (§ 81). A rotation-couple thus represents a displacement of the body, which cannot be regarded as a rotation. It is a vector q, whose absolute value is equal to the "moment" of the rotation-couple—that is, the product of the angle of rotation and the distance between the two axes of rotation, and whose direction is perpendicular to the plane of the rotation-couple that passes through it, in the sense determined by the two directions of rotation. This vector can be symbolized by a distance drawn from an end-point and provided with a double hook, as in Fig. 33, where the two rotations indicated in this way are also shown in perspective by dotted lines. In contrast with the symbol for a simple rotation, that for a rotation-couple can also be displaced laterally without its kinematic meaning being changed (§ 83). If we subject the body to several quite arbitrary rotation-couples $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, . . there will still always result a rotation-couple $\mathbf{q}$, which is obtained simply by vectorial addition (§ 84):

$$\mathbf{q} = \Sigma \mathbf{q}_1 \qquad \text{(344a)}$$

The simplicity of the properties of rotation-couples leads us also to surmise that a rotation-couple has a simple kinematic meaning. We can easily determine this by inquiring what is the displacement which the body undergoes when it is subjected to two equal anti-parallel rotations. If we take the one axis of rotation as the x-axis and the arm h of the rotation-couple as the y-axis

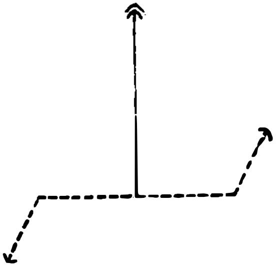

FIG. 33.

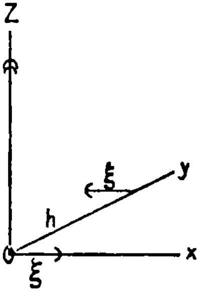

FIG. 34.

(Fig. 34), it is sufficient to calculate the displacement, which a point of the body originally situated in a co-ordinate-plane experiences through the two successive rotations. For a point of the xy-plane ($z = 0$), for example, the displacement is each time directed parallel to the z-axis—namely, for the one rotation $\delta_1 z = \xi y$, for the other $\delta_2 z = -\xi(y - h)$, so that together:

$$\delta z = \delta_1 z + \delta_2 z = \xi h \qquad (345)$$

Thus the displacement is everywhere of the same magnitude and in the same direction for all points of the xy-plane and hence also for all points of the body. Such a displacement of the body is called a *translation*. Hence we have quite generally the theorem that a rotation-couple $\boldsymbol{q}$ represents nothing else than a translation whose magnitude is by (345) the moment $|\boldsymbol{q}|$ of the rotation-couple and whose direction coincides with the axis of the rotation-couple.

The equation (344*a*) now acquires a new graphical meaning as regards the composition of rotation-couples in different directions, and likewise the theorem that the vector of a rotation-couple may also be displaced laterally. For, in a translation of a body, in contradistinction to a rotation or a force, all straight lines parallel to the direction of the vector are equivalent.

Furthermore, the following theorem (§ 85) holds. A rotation $\boldsymbol{o}$ about an axis which passes through any point $\boldsymbol{r}$ is kinematically equivalent to a rotation $\boldsymbol{o}$ about a parallel axis through the origin of co-ordinates, together with a translation :

$$\boldsymbol{q} = [\boldsymbol{r}, \boldsymbol{o}] \quad . \quad . \qquad . \quad (346)$$

which coincides in magnitude and direction with the displacement which the origin of co-ordinates undergoes through the originally assumed rotation (§ 87).

Finally, we may also answer generally the question proposed at the beginning of this section about the composition of arbitrary rotations (§ 88). If a free rigid body is subjected in any arbitrary order of sequence to an arbitrary number of infinitesimal rotations $\boldsymbol{o}_1$, $\boldsymbol{o}_2$, $\boldsymbol{o}_3$, . . . whose axes pass through the points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, . . ., the resultant displacement of the body is equivalent to a single rotation $\boldsymbol{o}$ about the origin of co-ordinates combined with a translation $\boldsymbol{q}$, where :

$$\boldsymbol{o} = \Sigma \boldsymbol{o}_1, \ \boldsymbol{q} = \Sigma[\boldsymbol{r}_1, \boldsymbol{o}_1] \quad . \quad . \quad . \quad (347)$$

Here $\boldsymbol{o}$ does not depend on the choice of the origin of co-ordinates, whereas $\boldsymbol{q}$ does.

§ 103. Before we proceed we must yet assure ourselves that the displacement (347) of the body is also the *most general* infinitesimal displacement which it can undergo at all. Actually, however the body may be displaced, the displacement can clearly always be produced by a translation which is so calculated that the origin of co-ordinates (or, more correctly, the material point which lies originally at the origin of co-ordinates) can be brought into its final position, and afterwards the body can be rotated about the origin of co-ordinates which is kept fixed. Whether this rotation follows about the origin of co-ordinates (at rest in space) or about the material point which lay at the origin of co-ordinates before the translation, causes only a vanishingly small difference in the result, since points which lie infinitely near the axis of rotation undergo displacements through the rotation which are only of a higher order of infinitesimals.

Having obtained this result, we may now use the principle of virtual work directly to derive the condition of equilibrium for a free rigid body. Let driving forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, . . . act at the points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, . . . of the body. We then suppose the body to be subjected to the most general displacement by means of a rotation $\boldsymbol{o}$ about the origin and a translation $\boldsymbol{q}$. By (339) the displacement of a point of the body is then :

$$\delta \boldsymbol{r}_1 = \boldsymbol{q} + [\boldsymbol{o}, \boldsymbol{r}_1] \quad . \quad . \quad . \quad . \quad (348)$$

where $\boldsymbol{q}$ and $\boldsymbol{o}$ remain without an index, since they are common to all points of the body. If this value is substituted in the expression (321) for the virtual work, we get after a little simplification the following condition for equilibrium :

$$\Sigma \boldsymbol{F}_1 \cdot \delta \boldsymbol{r}_1 = \boldsymbol{q} \cdot \Sigma \boldsymbol{F}_1 + \boldsymbol{o} \cdot \Sigma [\boldsymbol{r}_1, \boldsymbol{F}_1] = 0. \quad . \quad . \quad (349)$$

and since $\boldsymbol{q}$ and $\boldsymbol{o}$ are quite arbitrary, the equations (306*a*) follow, corresponding to the six degrees of freedom of the system.

§ 103*a*. We revert once more to the kinematic considerations of § 102 and pursue the analogy of the system of rotations with the system of forces a little further, in the sense of §§ 88 to 90. We may then immediately enunciate the following theorems.

The most general infinitesimal displacement of a free rigid body may also be represented as the result of two rotations whose axes do not intersect. Furthermore, it is always possible to choose the origin of co-ordinates O_0 so that the total displacement (347) of the body is represented by a single rotation $\boldsymbol{o}$ about O_0 and a translation $\boldsymbol{q}_0$ which falls in the direction of the axis of rotation. This special axis $\boldsymbol{o}$ which passes through O_0 is called the central axis of the rotations which produce the displacement (347). The corresponding translation $\boldsymbol{q}_0$ is the *smallest* among all the translations $\boldsymbol{q}$ which correspond to other origins O. Such a displacement ($\boldsymbol{o}$, $\boldsymbol{q}_0$) is called a "screw." Hence every infinitesimal displacement of a rigid body may be regarded as a screw. In special cases screw motion degenerates into a pure rotation ($\boldsymbol{q}_0 = 0$) or a pure translation ($\boldsymbol{o} = 0$).

§ 104. Hitherto we have always regarded the driving forces $\boldsymbol{F}$ as given and have made no detailed assumptions about their nature. It is obviously very important, however, to be able to make some general statements about these forces; we therefore proceed next to consider this aspect.

In the mechanics of a single material point we have already seen (§ 36) that if the resultant driving force $\boldsymbol{F}$ arises from central forces its components are the derivatives of a definite function, the negative potential $-U$, with respect to the co-ordinates of the point, or, what amounts to the same thing, the work of the driving forces forms the complete differential of $-U$. Precisely the same may be asserted in the case of any arbitrary system of points, both when the forces originate in fixed centres and when the moving points mutually act on one another with central forces.

To prove this assertion we reflect that the total work of the driving forces during any motion of the system of points is the sum of the amounts of work done by all the individual forces; we therefore consider the terms of this sum individually. Concerning the forces which arise from the fixed centres we at once recognize that the work done by them at each of the moving points 1, 2, 3, . . . is individually represented by a complete differential $-dU_1$, $-dU_2$, $-dU_3$, ..., as was proved in § 36. With regard to the mutual actions of the moving points we likewise imagine the total work resolved into the amounts of work done by the forces which two points exert on each other, for example, the points 1 and 2. This work is of the form :

$$X_{12}dx_1+Y_{12}dy_1+Z_{12}dz_1+X_{21}dx_2+Y_{21}dy_2+Z_{21}dz_2. \quad (350)$$

where the first of the two indices denotes the point on which the force-component acts, the second denoting the point in which it originates. If we denote the magnitude of the force by $f(r_{12})$ (calling it positive when it attracts), where :

$$r_{12}{}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \ . \quad (351)$$

then the equations (106) hold for the six force-components, and the expression (350) for the work becomes :

$$-\frac{f(r_{12})}{r_{12}}\Big\{(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) + (z_2 - z_1)(dz_2 - dz_1)\Big\} = -f(r_{12})dr_{12} = -dF(r_{12}). \quad (352)$$

according to the notation introduced in (107).

We therefore set the force-potential :

$$U = \underset{1}{\Sigma} U_1 + \underset{1,\,2}{\Sigma} F(r_{12}) \ . \quad . \quad . \quad (353)$$

where the summation is taken over all combinations of the points in pairs. Then, in any displacement of the points, the work performed by all the central forces is :

$$\Sigma \boldsymbol{F}_1 \cdot d\boldsymbol{r}_1 = -dU \quad . \quad . \quad . \quad . \quad (354)$$

and the negative derivative of U with respect to any co-ordinate represents the corresponding resultant force-component.

If we imagine the whole point-system to be sub-divided into two partial systems, then we see from equation (353) that the potential of the whole system is not equal, say, to the sum of the potentials of the individual systems, but that there must be added to these "self-potentials" of the two systems the "potential of the one system with respect to the other system." A corresponding result holds for the resolution into more than two systems.

§ 105. If we now apply the theorems of § 95 to the case where the driving forces have a potential U, we arrive at conclusions which represent a generalization of those already obtained in § 67. First, we have by (354) and (319):

$$dU < 0 \quad . \quad . \quad . \quad . \quad . \quad (355)$$

That is, if a system of points which is originally at rest and which is subject to any arbitrary prescribed conditions is set into motion by central forces the potential decreases in the process. We may also express this by saying: "the forces strive to diminish the potential." But if we have for every possible virtual displacement:

$$\delta U \geqq 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (356)$$

in the sense of (320), then a state of equilibrium exists. For if there is no possible way by which the forces can diminish the potential, they cannot effect a change in the position of the point-system.

We may go still further. If in a certain position of the point-system the function U has the smallest value which it can assume at all in view of the prescribed conditions, then this system is in stable equilibrium in this position. For not only does the condition of equilibrium $\delta U = 0$ then hold, but the system also reverts, when slightly disturbed from its position of equilibrium and then left to itself, to its position of equilibrium, by (355), since the potential cannot be diminished by another displacement. Conversely, the equilibrium is unstable when U is a maximum, because the point-system, when disturbed from this position, is unable, by (355), to return to it. But if U is quite independent of the co-ordinates of the points, then likewise $\delta U = 0$, and hence there is also equilibrium;

but this equilibrium is neutral—that is, it exists for every position of the points.

§ 106. As a simple example of the theorems here derived, we consider a system of discrete or continuously distributed heavy point-masses m_1, m_2, m_3, . . . which are subject to arbitrary prescribed conditions, being, say, partly connected with one another rigidly or being fixed, and so forth. Since gravitation is a central force, the driving forces here have a potential which, if the z-axis is taken as vertical, is obtained according to (354) and (76a), from the equation :

$$\Sigma \boldsymbol{F}_1 \,.\, d\boldsymbol{r}_1 = -g\Sigma m_1 dz_1 = -dU$$

and by (287) :

$$U = gz_0 \,.\, \Sigma m_1 + \text{const.} \quad . \qquad (357)$$

That is, the gravitational potential of a point-system is, except for an additive constant of no importance, the product of the acceleration due to gravity, the total mass of the system and the height of the centre of gravity. Since in this product the height z_0 of the centre of gravity is the only variable, we obtain, according to the preceding section, the general theorem that every transition of such a point-system from a state of rest to a state of motion is accompanied by a lowering of the centre of gravity, and that a maximum, a minimum, or no change in the height of the centre of gravity denotes unstable, stable or neutral equilibrium. In many cases we are able to see at once that this theorem is correct, as, for example, in the case of a rigid body with a fixed axis about which it may rotate; here the centre of gravity lies above, below or at this fixed point. But in other cases it leads to consequences which do not appear evident at the outset. For example, if we suspend a heavy chain of any kind, with equal or unequal links, from two fixed points and allow it to hang freely between them, then when the chain is in stable equilibrium it always assumes of all possible positions that for which its centre of gravity lies lowest. From this condition it is possible to calculate the form of the chain when in equilibrium.

§ 107. Having dealt with three-dimensional rigid bodies, we shall next consider another example of a partially non-free system of points—namely, *an inextensible but perfectly elastic thread or string.* Such a system is represented by a singly infinite series of one-dimensional—that is, linearly infinite—small rigid elements of mass, each of which is connected at its extremities with the preceding and the following element in a manner allowing of free rotation. Only the starting-point and the end of the string are unconnected and subject to special conditions. Let us suppose that given driving forces act on the string; we shall take them to be distributed continuously over its elements, assuming that the force which acts on the element of arc ds of the string has the components :

$$\mathbf{F}_x ds, \mathbf{F}_y ds, \mathbf{F}_z ds \quad . \quad . \quad . \quad . \quad (358)$$

Suppose the two extremities of the string to remain fixed. Our object is to find the position of equilibrium of the string.

We shall discuss this problem, too, according to the two methods with which we have become acquainted, each of which has its particular advantages : the first introduces constraining forces, the second uses the principle of virtual work.

Concerning the forces of constraint, which cause the string to be inextensible, those acting at a point P of the string are defined by the circumstance that they represent the forces which must be applied at the point P in order that the mechanical state of the point-system in question shall in no way be disturbed if we imagine the string to be severed at this point. Obviously we must apply two forces at P for this purpose, which represent the constraining forces with which the two elements of string that are adjacent at P act on each other. According to Newton's third law (principle of action and reaction) these forces are equal in magnitude and opposite in direction, and are called the "tension" S of the thread at the point P.

The magnitude of S will in general vary from point to point. Since the constraint opposes only the lengthening and not the bending of the string, the direction of S coincides at every point with the direction of the tangent to the curve of the string.

Equilibrium will occur when every element of the string is in equilibrium. Let us consider such an element PQ of the string, its length being ds (Fig. 35). Three forces act on such an element: (1) the tension S at the point P, (2) the tension $S + dS$ at the point Q, (3) the driving force $\boldsymbol{F} . ds$. We form the x-components of these three forces and set their sum equal to zero. The tension at P acts tangentially on the element of the string in the direction of decreasing s, if we set:

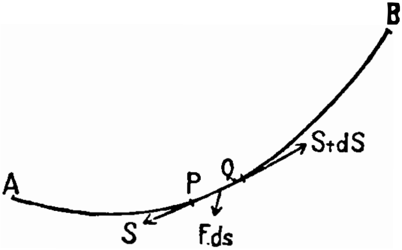

FIG. 35.

$$s = 0 \text{ and } s = l \text{ (length of the string)} \tag{359}$$

at the extremities A and B of the whole string: thus the required component is: $-S\frac{dx}{ds}$. The tension at Q has a different value and a different direction from that at P, in addition to the difference in sign. Hence its x-component is:

$$\left(S\frac{dx}{ds}\right)_{s+ds} = S\frac{dx}{ds} + \frac{d}{ds}\left(S\frac{dx}{ds}\right) \cdot ds.$$

By adding up the three components and omitting the common factor *ds* we get :

$$\left.\begin{aligned} &\frac{d}{ds}\left(S\frac{dx}{ds}\right)+F_x=0\\ \text{Similarly :}\qquad &\frac{d}{ds}\left(S\frac{dy}{ds}\right)+F_y=0\\ \text{and :}\qquad &\frac{d}{ds}\left(S\frac{dz}{ds}\right)+F_z=0 \end{aligned}\right\} . \qquad (360)$$

This solves the problem in every respect. For these three equations not only give us the two equations for the equilibrium curve of the string, when we eliminate S, but they also give us the value of the tension at every point of the string. No special condition of equilibrium is required for the extremities A and B of the string, since these are fixed points.

§ 108. We shall now solve the same problem by using the principle of virtual work. For this purpose we must set up the most general expressions for the virtual displacements of all the points x, y, z of the string. Since all the individual elements of length are rigid, the variation of ds or of $ds^2 = dx^2 + dy^2 + dz^2$ is equal to zero, thus :

$$dx\,.\,\delta dx + dy\,.\,\delta dy + dz\,.\,\delta dz = 0 \quad . \quad . \quad (361)$$

In order to bring out the sense of these expressions clearly, we may suppose that the co-ordinates x, y, z besides depending on the parameter s, also depend on a second parameter p chosen quite arbitrarily. Corresponding to a definite value of p there is then a definite curve for the string, and for a changed value $p + \delta p$ there is a definite infinitely near curve which represents the positions of the varied points. At the same time, $\delta x = \frac{\partial x}{\partial p}\delta p$, and so forth. The operations d and δ which correspond to the changes ds and δp are entirely independent of each other and may therefore be commutated—that is, $\delta dx = d\delta x$, and so forth.

The equation (361) represents an infinite number of fixed conditions of the form (322). We apply to them Lagrange's method of elimination described in § 96, by multiplying them by factors which are left undetermined initially and which vary from equation to equation, then adding them together and to the virtual work, and treating the variations δx, δy, δz as independent of one another. In this way we get from (361):

$$\int_0^l \left(\frac{dx}{ds}\cdot\frac{d\delta x}{ds}+\frac{dy}{ds}\cdot\frac{d\delta y}{ds}+\frac{dz}{ds}\cdot\frac{d\delta z}{ds}\right)\cdot\lambda\cdot ds=0,$$

where λ may be a perfectly arbitrary function of s; by adding this expression to the virtual work of the driving forces (358) we get:

$$\int_0^l \left(\boldsymbol{F}_x\delta x+\lambda\frac{dx}{ds}\cdot\frac{d\delta x}{ds}+\ .\ .\ .\right)ds=0.$$

To reduce this expression to terms of the independent variations δx, δy, δz, we must change its form, since δx does not occur explicitly in the term containing λ, but differentiated with respect to s. This may be done by integrating by parts:

$$\int_0^l \lambda\frac{dx}{ds}\cdot\frac{d\delta x}{ds}\cdot ds=\left[\lambda\frac{dx}{ds}\cdot\delta x\right]_0^l-\int_0^l\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)\delta x\cdot ds\ . \qquad (362)$$

Here the first term on the right-hand side vanishes because the extremities of the string are fixed. Hence we obtain as the condition for equilibrium:

$$\int_0^l \left(\boldsymbol{F}_x-\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)\right)\delta x ds+\ .\ .\ .=0$$

and by setting the coefficients of all the individual variations for each element of the string equal to zero we get:

$$\boldsymbol{F}_x-\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)=0,\ .\ .\ .\ .\ \qquad (363)$$

If we eliminate λ from these three equations we clearly obtain the same curve as in (360), and so the results are in agreement.

The advantage of using the principle of virtual work consists, as always, in the fact that this method is independent of particular mechanical considerations. For the same reason, however, it gives no insight into the mechanical conditions. For the physical meaning of λ emerges only when we compare the equations with equations (360), which show that λ is the negative tension.

§ 109. We now draw conclusions of a general kind from the equations (360). If we write them in the form:

$$\frac{dS}{ds}\frac{dx}{ds} + S\frac{d^2x}{ds^2} + \boldsymbol{F}_x = 0, \quad . \quad . \quad . \quad . \quad . \quad (364)$$

multiply them by the direction-cosines of the bi-normals (§ 25) of the curve and add up, we see that the direction of $\boldsymbol{F}$ lies in the plane of curvature of the curve of the string—which we also find necessary from simple physical considerations.

But if we multiply them by the direction-cosines of the element of arc ds and add, we get by (73) and (73*a*):

$$\frac{dS}{ds} + \boldsymbol{F}_x\frac{dx}{ds} + \boldsymbol{F}_y\frac{dy}{ds} + \boldsymbol{F}_z\frac{dz}{ds} = 0 \quad . \quad . \quad (365)$$

That is, the change of the tension along the string is measured by the component of $\boldsymbol{F}$ in the direction of the string. If the force $\boldsymbol{F}$ is everywhere perpendicular to the curve of the string, the tension is everywhere the same.

If the force $\boldsymbol{F}$ has a potential, then:

$$\frac{dS}{ds} - \frac{dU}{ds} = 0, \; S = U + \text{const.} \quad . \quad . \quad (366)$$

That is, the tension is equal to the potential (referred to unit length) except for an additive constant.

§ 110. Let us next assume that the driving force is gravity and that the string is homogeneous; its form when in equilibrium is called the *catenary* (*Kettenlinie*). If the mass of the whole string is M, that of the element

ds of the string is equal to $M \cdot \frac{ds}{l}$, and the components (358) of the force which acts on it are:

$$\boldsymbol{F}_x = 0,\ \boldsymbol{F}_y = 0,\ \boldsymbol{F}_z ds = -M \cdot \frac{ds}{l} \cdot g\,. \qquad . \quad (367)$$

Hence the potential of the force $\boldsymbol{F}$ is:

$$U = \frac{M}{l} gz + \text{const.} \quad . \quad . \qquad . \quad (368)$$

By § 109 the curve of the string lies in a vertical plane which is determined by the extremities A and B of the string. If we choose this plane as our xz-plane, the equations (360) reduce to:

$$\frac{d}{ds}\left(S\frac{dx}{ds}\right) = 0,$$

$$\frac{d}{ds}\left(S\frac{dz}{ds}\right) - \frac{Mg}{l} = 0.$$

Integration gives:

$$S\frac{dx}{ds} = \frac{Mg}{l} c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (369)$$

$$S\frac{dz}{ds} = \frac{Mg}{l}(s + c_1) \quad . \qquad . \quad (370)$$

where c and c_1 represent two constants having the dimensions of a length.

The following relationship, which results from (366) and (368), is also of value:

$$S = \frac{Mg}{l}(z + c_2) \quad . \quad . \quad . \quad . \quad . \quad (371)$$

To get the equation to the curve in rectilinear coordinates we find it convenient to eliminate S from (369) and (371):

$$\frac{dx}{ds} = \frac{c}{z + c_2}.$$

By substituting $ds^2 = dx^2 + dz^2$ we get the differential equation :

$$dx = \frac{cdz}{\sqrt{(z + c_2)^2 - c^2}},$$

the integral of which is :

$$x = c \log \left\{ \frac{z + c_2}{c} + \sqrt{\left(\frac{z + c_2}{c}\right)^2 - 1} \right\} + c_3 \quad . \qquad (372)$$

Solving for z we have :

$$z = \frac{c}{2}\left(e^{\frac{x - c_3}{c}} + e^{-\frac{x - c_3}{c}}\right) - c_2 \quad . \qquad (373)$$

The values of the four constants c, c_1, c_2, c_3 are given by the two equations (359) and the two conditions that the given points A and B lie on the curve.

The equation to the catenary assumes its simplest form if we take as the origin of co-ordinates the point $x = c_3$, $z = - c_2$. It then runs :

$$z = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \quad . \qquad (374)$$

The curve runs symmetrically on both sides of the z-axis ; as x increases or decreases it rapidly ascends and has its minimum at the point $\frac{dz}{dx} = 0$, or

$$x = 0, \ z = c.$$

Instead of the points A and B we may of course take any other two arbitrary points of the curve, on the same side or on the opposite side of the minimum, as the fixed points of suspension without changing the form of the curve. The tension S then simply becomes by (371) :

$$S = \frac{Mg}{l} z \qquad (375)$$

and its minimum value is thus $\frac{Mg}{l} c$.

§ 111. We shall next consider the equilibrium of a string which is stretched over a fixed surface $f(x, y, z) = 0$, and shall begin by considering the special case where a driving force $\boldsymbol{F}$ acts only on the extremity B of the string. This may be imagined to be realized by having the string fixed at the point A of the surface and to be drawn through a small ring fixed at the point B of the surface, where it is held taut by the force $\boldsymbol{F}$. The fixed surface of course presents its convex surface to the string, for otherwise the string would not lie on the surface at all. The equations (360) then become :

$$\frac{d}{ds}\left(S\frac{dx}{ds}\right) + \boldsymbol{Z}_x = 0, \ldots \quad . \qquad (376)$$

where the components of the force of constraint $\boldsymbol{Z}$ which is exerted by the fixed surface on the unit of length of the string satisfy the condition (246). From this and from the equation to the surface $f = 0$, or also in the differential form :

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = 0 \quad . \qquad (377)$$

we get the equation to the curve of the string, as well as the tension S of the string and the force of constraint $\boldsymbol{Z}$ that acts on it which, by Newton's third law, also represents the pressure which the surface experiences owing to the stretched string.

If we multiply the equations (376) by $\frac{dx}{ds}$ and so forth, and add up, we get, in view of (246) and (377) :

$$\frac{dS}{ds} = 0, \; S = \text{const.} \quad . \quad . \quad . \qquad (378)$$

That is, the tension of the string is everywhere the same and is equal to the value F of the driving force, since it balances this force at B—a result which emerges directly from (365) if we consider that the force of constraint $\boldsymbol{Z}$ is everywhere perpendicular to the curve of the string. So the equations (376) become simplified to :

$$F \cdot \frac{d^2x}{ds^2} + \boldsymbol{Z}_x = 0, \ldots \qquad (379)$$

from which we get the value of the constraining force as :

$$|\boldsymbol{Z}|^2 = F^2 \cdot \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}$$

or, by (74) :

$$|\boldsymbol{Z}| = \frac{F}{\rho} \quad . \quad . \quad . \quad . \quad . \quad (380)$$

That is, the value of the pressure exerted by unit length of the string on the surface is equal to the quotient of the stretching force and the radius of curvature of the curve of the string. The more curved the string, the greater this pressure; for a straight line it vanishes entirely.

Since the direction of $\boldsymbol{Z}$ is given by (246), the equations (250) follow from (379) for the curve of the string; these equations state that when the string is in equilibrium it assumes the form of a geodetic line on the surface; this line passes through the points A and B.

In view of the principle of virtual work a new and important property of geodetic lines follows from this. According to this principle the string is in stable equilibrium when, of all the curves that can be drawn on the surface from A to B it assumes that which is of shortest length. For otherwise the driving force $\boldsymbol{F}$, the only force which is present at all, would be able to perform positive work by drawing the string through the ring B (§ 95). Hence every curve of shortest length on the surface is at the same time a geodetic line of the surface. It is to this property that geodetic lines owe their name, since the distance between any pair of points on the earth's surface is measured by the shortest connecting line.

On a sphere the shortest lines are the great circles; on a plane they are the straight lines.

But the converse of this theorem is not in general true. That is, a geodetic line is not always the shortest line between two of its points, just as the equation $\delta s = 0$ is indeed the necessary but not always at the same time the sufficient condition, that s should be the minimum. For example, an arc of a great circle on a sphere actually loses the property of being the shortest connecting line between its extremities if the length of the arc is greater than the semicircumference. A stretched string which lies along it is then in equilibrium, but this equilibrium is no longer stable.

§ 112. In all our discussion so far we have assumed the driving forces to be given. But in nature we often have to do with problems in which driving forces of a complicated kind difficult to define come into play, particularly when they act in the interior of the bodies. It is therefore of very great importance to have a principle which leads to a simple and conveniently applicable condition of equilibrium even in the most complicated cases. To derive this principle we revert again to the arguments brought forward in the introduction to the present part of this volume—namely, § 76. On the view there described we may divide all active forces into *internal* and *external* forces. Internal forces are all those which arise from points of the system, external forces are all those which arise from points outside the system. The question as to whether a certain force on which we fix our attention is an internal or an external force may accordingly be decided only when we have made our choice of the system of points, this choice being quite arbitrary at the outset. In this way we can convert every internal force into an external force by excluding the point at which it arises from the system, and conversely.

This division into internal and external forces is not of course coincident with that into driving forces and forces of constraint. There are internal and external driving forces, and there are internal and external forces of constraint. In the case of a heavy rigid body with a fixed point of rotation, for example, the molecular forces are internal forces of constraint. The support of the fixed point is an external force of constraint, the gravitational force is an external driving force. But if we include the earth in the system of points all these forces become internal forces.

What makes the distinction between internal and external forces so fruitful is the circumstance that the internal forces in a point-system always occur in pairs, and, moreover, so that they are equal in magnitude but opposite in sign (§ 76).

This circumstance combined with the theorem—the truth of which we immediately recognize—that every state of equilibrium of every point-system remains preserved as such if we imagine all the points of the system to be rigidly connected together, leads to the fundamental theorem : *if a point-system is in equilibrium the external forces maintain equilibrium among themselves when acting on the system which we suppose rigid.* For since the internal forces acting on the rigid system cancel each other in pairs, they may be omitted altogether. This gives rise to a considerable simplification which is the more appreciated since it is just the internal forces that are very little known in many cases.

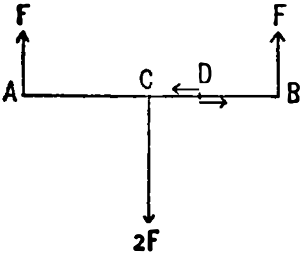

FIG. 36.

Since the choice of the point-system is perfectly arbitrary, the principle just mentioned entails numerous consequences, the abundance of which may be demonstrated in a few special examples.

§ 113. Let us take the simple case of a rigid rectilinear rod which is in equilibrium under the action of two equal parallel forces F at its end-points A and B, and an anti-parallel force $2F$ at its mid-point. We now select a part of the rod, say AD (Fig. 36), as a point-system. Then the external forces consist of the force F at A, the force $2F$ at C, and the forces with which the part of the rod DB at D acts on the part AD. Since the external forces together maintain equilibrium among themselves, the action of the part DB on the part AD consists of the force F with the point of application D together with a couple of moment $F \, . \, BD$, whose axis is perpendicular to the plane of the diagram (in Fig. 36 this direction is from the diagram towards the observer).

This couple can be realized only by different forces acting at different points of the cross-section of the rod at *D*. (In Fig. 36 the forces acting on the upper half are from right to left; on the lower half they are from left to right, as is indicated by the small arrows.) An infinitely small cross-section or a flexible string would not be able to achieve this; but in a finite cross-section there is a pressure on one side and a tension on the other.

In this way our principle gives us information about the force conditions that obtain in the interior of the body. Corresponding to the action of one part of the body on another as above considered there is of course always the equal and opposite action of the second part on the first.

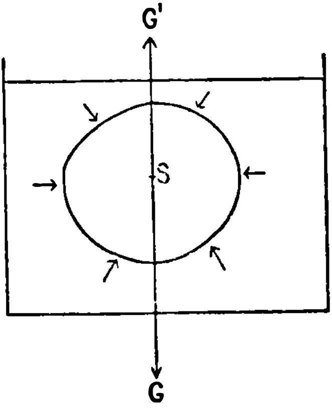

Fig. 37.

§ 114. We take another example from the realm of fluids. Let us consider a large quantity of some heavy liquid in a state of rest and let us choose a part of it of arbitrary shape, which is surrounded on all sides by liquid, as a point-system, then the external forces are: the weight of the liquid system, G_f, and the pressures exerted by the adjacent parts of the liquid on the system (in Fig. 37 these are indicated by arrows) with the resultant F. Hence, by the theorem of § 112, we have:

$$F = -G_f \quad . \quad . \quad . \quad . \quad . \qquad (381)$$

The pressures give a resultant which is equal and opposite to the weight of the liquid system and is called the "upthrust."

Let us next suppose that in place of the point-system in question we have any rigid system of exactly the same form, which we assume to be heavier than the liquid and which is prevented from sinking by being suspended by a string. If we take this rigid body as a point-system, then the external forces are its weight G, the pressures of the adjacent liquid with the resultant F, and the upward pull of the string, which gives the "apparent weight" G' of the body in the liquid. Accordingly we have:

$$G + F - G' = 0$$

or, by (381):

$$G' = G - G_f \quad . \quad . \quad . \quad . \quad . \qquad (382)$$

That is, the apparent weight of the body in the liquid is equal to its real weight diminished by the weight of the liquid displaced; this is Archimedes' Principle.

§ 115. Finally we apply our principle to a gaseous body—namely to the equilibrium of the atmosphere. Let us consider a vertical cylinder of air of unit cross-section and let us imagine a layer of air to be cut out of it by means of two horizontal cross-sections; we take this portion as a point-system. The external forces are then, first, the weight of the layer of air, secondly, the pressure of the surrounding air which acts downwards at the upper cross-section, upwards at the lower cross-section, and horizontally inwards on the curved surface of the cylinder. Our principle § 112 then demands that the weight of the layer of air be equal to the difference of pressure at the lower and the upper cross-section.

If we assume the layer of air to be infinitely thin, then its weight is proportional to the density of the air at the height in question, and by introducing the general relationship between density and pressure we get the differential equation which enables us to calculate the decrease of the pressure of the air as we ascend.

So we see how fundamental laws in hydrostatics and aerostatics emerge from the general principle of equilibrium for rigid bodies.

CHAPTER III

DYNAMICS OF AN ARBITRARY POINT-SYSTEM

§ 116. We are now sufficiently prepared to develop the general laws which contain as special cases the laws of the mechanics of a single material point and the laws of statics of any arbitrary point-system.

Suppose we have to determine the motion of a system of n material points of mass m_1, m_2, . . . on which the driving forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, . . . act, and whose freedom of motion is restricted by p equations of condition $f = 0$, $\phi = 0$, . . . between the co-ordinates of the points and the time t.

The solution of this problem is obtained directly by applying the principle of d'Alembert (§ 66), according to which the point-system is in equilibrium at any moment of time t, if we suppose the inertial resistances $-m_1\dot{\boldsymbol{q}}_1$, $-m_2\dot{\boldsymbol{q}}_2$, . . . acting at the individual points to be added to the forces acting on those points.

Thus with one stroke this problem of dynamics is con verted into a problem of statics, and we may immediately apply the principle of virtual work (321):

$$\Sigma(\boldsymbol{F}_1 - m_1\dot{\boldsymbol{q}}_1) \,.\, \delta\boldsymbol{r}_1 = 0 \quad . \quad . \quad . \quad (383)$$

or the equations (324) of Lagrange:

$$\boldsymbol{F}_{x_1} - m_1\frac{d^2x_1}{dt^2} + \lambda\frac{\partial f}{\partial x_1} + \mu\frac{\partial \phi}{\partial x_1} + \ . \ . \ . = 0 \ . \quad (384)$$

and so forth for all co-ordinates and all points. If we multiply the equations (384) individually by the corresponding co-ordinate variations δx_1, . . . and add, we again get (383), if we take into consideration the equations of condition (322) for the variations.

The elimination of the p quantities λ, μ, . . . from (384) gives us $3n - p$ linear equations between the accelerations and the driving forces, which, when taken with the p prescribed conditions, enable us to calculate the accelerations uniquely.

§ 117. A special remark must be made about the case where the time t occurs explicitly in the equations of condition $f = 0$, . . ., as, for example, when a point is constrained to move on a surface which is itself compelled to move in a certain way. We might be doubtful from the very outset as to what condition the virtual displacements must satisfy, since the equation of condition contains the variable parameter t. The equations (322), which, according to our above discussion, must also be valid here, show that when the co-ordinates are varied the time t remains unvaried—that is, that, for example, in the case of a point situated on a moving surface, the virtual displacement at the time t is of the same kind as if the surface is at rest in the position which it occupies at the time t.

A concrete idea of the meaning of this circumstance is given by the case treated in § 75 of a straight line which is rotated with constant angular velocity ω, as depicted in Fig. 17. Here the virtual displacement must be taken along the straight line which is at rest at the time t —that is, in the direction AB and not in the direction AA'; it is only then that the virtual work of the constraining force is equal to zero. Consequently we have from the equation of condition (277), when subjected to variation with t constant:

$$\delta y = \tan(\omega t) \, . \, \delta x,$$

and this combined with the principle (383):

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y = 0$$

gives us the same equation of motion (278a) as before.

§ 118. The equation (383) holds for every arbitrary system of infinitesimal co-ordinate displacements δx_1, . . . which satisfies the conditions (322). If by way of contrast we consider the infinitesimal co-ordinate displacements dx_1, . . . which actually occur in the element of time dt when the points move, we see that they satisfy the conditions :

$$\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial y_1}dy_1+\frac{\partial f}{\partial z_1}dz_1+\frac{\partial f}{\partial x_2}dx_2+\ \ldots\ +\frac{\partial f}{\partial t}dt=0\ . \qquad (385)$$

which are definitely distinguished from the equations (322) in that they contain in addition the terms in $\frac{\partial f}{\partial t}$, $\frac{\partial \phi}{\partial t}$, . . . Hence in general the true displacements dx_1, . . . do not belong to the system of virtual displacements, and we are not allowed to replace $\delta \boldsymbol{r}$ in (383) by $d\boldsymbol{r}$.

But if the prescribed conditions $f = 0$, $\phi = 0$, . . . do not contain the time explicitly, as we shall now assume, the terms which constitute the difference between (322) and (385) vanish, and the actual displacements become a special case of the virtual displacements. Hence by (383) we also have :

$$\Sigma(\boldsymbol{F}_1 - m_1\dot{\boldsymbol{q}}_1) \,.\, d\boldsymbol{r}_1 = 0,$$

or, otherwise expressed :

$$dK = \Sigma \boldsymbol{F}_1 \cdot d\boldsymbol{r}_1 \qquad . \quad . \quad (386)$$

where we set :

$$K = \frac{1}{2}\Sigma m_1 q_1^2 = \frac{1}{2}\Sigma m_1(u_1^2 + v_1^2 + w_1^2) \quad . \quad (387)$$

If we call the quantity K the kinetic energy or the vis viva of the point-system, the equation (386) states that *the change in the kinetic energy of the point-system is equal to the total work of the driving forces*; this is completely analogous to equation (147) and is quite independent of the prescribed conditions, as in § 67.

This simple relationship arises of course from the fact that the total work of the forces of constraint vanishes not only for every virtual displacement, but also for every true displacement of the point-system in the element of time *dt*. This would not be the case if the prescribed conditions contained the time *t* explicitly, as was explained in detail in the simple example of § 75.

§ 119. For forces which have a potential U, we have by (354) and (386) if we integrate with respect to the time t:

$$K + U = \text{const.} = K_0 + U_0 \, . \quad . \quad . \quad (388)$$

and if we again, as in § 49, call the quantity:

$$K + U = E \quad . \quad . \quad . \quad . \quad . \quad (389)$$

the *energy* of the point-system—that is, the sum of the *kinetic energy K* and the *potential energy U*—then the equation (388) states *the law of conservation of energy.* We have already discussed in detail in § 49 how it is to be generalized for non-mechanical processes.

The equation (388) also makes it possible for us to make a further deduction from the arguments contained in § 105. It was there shown that corresponding to a minimum of the potential U there is a stable state of equilibrium of the point-system. We proved this by reflecting that when the point-system is slightly disturbed from its position of equilibrium and then left to itself at rest, it can only move in a direction leading to the minimum of U. Now, a slight disturbance of the equilibrium can also be produced in a more general way—namely, by imparting to the points a small initial velocity before they are left to themselves.

In the initial state of the motion which then occurs K_0 is a small positive quantity, and U_0 is equal to $U_{\text{min.}} + U'_0$, where U'_0 is likewise small and positive. Consequently we have by (388) for the whole duration of the motion that:

$$K + U - U_{\text{min.}} = K_0 + U'_0 \, . \quad . \quad . \quad (390)$$

is small and positive. Since K consists only of positive terms and since $U - U_{\text{min}}$ is positive, the velocities of all the points are permanently small, and the point-system persists near its position of equilibrium—that is, the equilibrium is stable. A corresponding argument holds for a maximum of U.

§ 120. In general, the driving forces which act on the point-system are not of a conservative nature (§ 49), particularly not when more or less arbitrary disturbances are introduced from outside. Hence we shall now assume that the driving forces are of two kinds: conservative forces and external forces of a non-conservative kind which we denote by $\boldsymbol{F}_a$. Then we have in general for the work of the driving forces:

$$\Sigma_1 \boldsymbol{F}_1 d\boldsymbol{r}_1 = -dU + A \quad . \quad . \quad . \quad (391)$$

where we write:

$$A = \Sigma \boldsymbol{F}_a \cdot d\boldsymbol{r} \quad . \quad . \quad . \quad . \quad . \quad (392)$$

for the work of the external forces or the " external work." The equation (386) of energy then becomes:

$$d(K + U) = dE = A \,. \quad . \quad . \quad . \quad (393)$$

That is, the change of energy of the point-system is equal to the external work, being positive or negative according as the external work is done " on " or " by " the system. In the former case the change occurs in the sense of the external forces, in the latter in the opposite sense.

If we include the points or bodies from which the external effects emerge in the point-system under consideration, all the external forces vanish (cf. § 112), and the system is called a " complete " or a " closed " system. For a closed system the energy-principle again holds in the form (388) as the law of conservation of energy, and in this sense we speak of the conservation of energy of the whole world as of that material system which comprises all bodies capable of producing effects. We must bear in mind, however, that in nature a closed system in the absolute sense cannot be demonstrated with certainty, and hence that we cannot calculate with the energy of the " world " as a definite quantity.

This does not, of course, prevent us in some circumstances from treating even arbitrarily small finite point-systems, if sufficiently isolated, as closed systems.

If we resolve a complete system into two partial systems, then the work done by the points of the one partial system on those of the other will entail a change in the energies of the partial systems—that is, through this work energy will be transferred from the one partial system to the other, whereas the total energy will remain constant. We must take care to note, however, that in general the potential energy of the whole system is not, like the kinetic energy, equal to the sum of the energies of the partial systems (§ 104).

§ 121. To obtain a clear idea of the magnitude of the kinetic energy of a point-system it is often found expedient to refer it to a moving co-ordinate-system, whose origin is at the centre of gravity of the system. Then the equations of transformation (191) give us the following expression for the value (387) of the kinetic energy :

$$K=\frac{1}{2}\Sigma m_1 q'^2_1+u_0\Sigma m_1 u'_1+v_0\Sigma m_1 v'_1+w_0\Sigma m_1 w'_1+\frac{1}{2}q_0^2\Sigma m_1.$$

But since, as we can find by differentiating (287) with respect to the time :

$$\Sigma m_1 u'_1 = \Sigma m_1(u_1 - u_0) = 0, \quad . \quad . \quad . \quad . \quad (394)$$

the kinetic energy reduces to :

$$K = \frac{1}{2} q_0^2 \Sigma m_1 + \frac{1}{2} \Sigma m_1 q'^2_1 \quad . \quad . \quad (394a)$$

That is, the kinetic energy of a point-system is composed additively of the kinetic energy of its centre of gravity, if we suppose all the masses to be concentrated in it (energy of " translation ") and the kinetic energy relative to the centre of gravity (energy of " vibration," which includes motions of rotation as special cases).

§ 122. The fundamental law of mechanics which we have hitherto expressed in the equations (383) or (384) may be formulated in several other ways, which have exactly the same physical content, but in their applications emphasize very different features. The most important of these is the *Principle of Least Action.* We shall develop it here in the form given by Hamilton.

Since the equation (383) of the principle of virtual work holds for any time t, we may also integrate it with respect to t between the limits t_0 and t_1, and so obtain :

$$\int_{t_0}^{t_1} dt \,.\, \Sigma\Big(\boldsymbol{F}_x - m\frac{d^2x}{dt^2}\Big)\delta x = 0 \,. \qquad (395)$$

where the summation is to be taken over all co-ordinates and all points of the system. Here not only the co-ordinates x, y, z, but also the variations δx, . . . are to be treated as functions of the time t. To see this clearly it is best to imagine the co-ordinates of all the points, besides depending on t, also to depend on a second parameter p which is selected quite arbitrarily and which is finite, as has already been done in § 108. Corresponding to a definite value of p, there is the required motion of the point-system, for the present still unknown, and corresponding to a changed value $p + \delta p$ there is another definite motion "infinitely adjacent" to the real motion, but which does not satisfy the equations of motion. The operations d and δ, which correspond to the changes dt and δp, are entirely independent of each other, and may therefore be commuted :

$$\frac{d\delta x}{dt} = \delta\frac{dx}{dt} \quad . \quad . \qquad . \quad . \qquad (396)$$

The variations $\delta x = \frac{\partial x}{\partial p}\delta p$, . . . are quite arbitrary at every moment of time t and are subject only to the conditions (322), in which the functions f, ϕ, . . . may also contain the time t explicitly.

By writing the time-integral (395) in a somewhat different form, we next obtain for the virtual work, by (391) :

$$\Sigma \boldsymbol{F}_x\delta x = -\,\delta U + A \quad . \qquad . \qquad (397)$$

where U denotes the potential of the conservative forces of the system, and A the virtual work of the external forces. An external force, which is at the same time conservative (such as gravity), can be included either in $-\delta U$ or A according to our wishes.

Further, integration by parts gives us the term :

$$\int_{t_0}^{t_1} dt \cdot \frac{d^2x}{dt^2}\delta x = \left[\frac{dx}{dt}\delta x\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} dt \cdot \frac{dx}{dt}\frac{d\delta x}{dt} \quad . \qquad (398)$$

and, if we now introduce the assumption that the variations of the co-ordinates of all the points vanish for $t = t_0$ and $t = t_1$, then in view of (396) :

$$\int_{t_0}^{t_1} dt \cdot \frac{d^2x}{dt^2}\delta x = -\frac{1}{2}\delta\int_{t_0}^{t_1} dt \cdot \left(\frac{dx}{dt}\right)^2 \quad . \quad . \qquad (399)$$

and hence by substituting in (395), by (387) :

$$\int_{t_0}^{t_1} dt\,(\delta L + A) = 0 \quad . \quad . \quad . \quad . \qquad (400)$$

where we have used the abbreviation :

$$L = K - U \quad . \quad . \quad . \quad . \quad . \qquad (401)$$

The equation (400) expresses Hamilton's principle of least action. The function L, which is not to be confused with the energy E, is called *Lagrange's function* or the *kinetic potential.* In contrast with d'Alembert's principle, according to which the motion is determined by the initial positions and initial velocities of the points, the motion here, by the principle of least action, is determined solely by the initial positions ($t = t_0$) and the final positions ($t = t_1$) of the points. For it is these points which are kept fixed in all the infinitely near motions that are considered, whereas the velocities, including the initial velocities, may be varied arbitrarily within the range of the prescribed conditions.

The supreme importance of the principle of least action for the whole of physics depends on the fact that the concepts of potential and external work that occur in it also have a meaning outside mechanics and on the further fact that the principle has not been shaped for use with a definite kind of co-ordinates. These circumstances enable it to be applied directly also to electrodynamic and thermodynamic processes, where it has everywhere been of great value.

§ 123. Let us consider a simple application. How does a material point subject to no driving forces move on a fixed surface? By (400) and (401) we have for this case:

$$\int_{t_0}^{t_1} dt \, . \, \delta K = 0$$

or:

$$\delta \int_{t_0}^{t_1} K \, . \, dt = 0 \qquad . \qquad (402)$$

Expressed in words: among all the motions possible on the surface, which bring the point from a definite initial position in a definite time $t_1 - t_0$ into a definite final position, that motion actually occurs in nature which makes the time-integral of the kinetic energy a minimum.

This single theorem gives us both the form of the orbital curve and the velocity with which it is traversed. For if we substitute the value:

$$K = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2$$

we get from (402):

$$\delta \int_{t_0}^{t_1} \left(\frac{ds}{dt}\right)^2 \cdot dt = 0,$$

or, if we perform the variation and then integrate by parts:

$$\left[\frac{ds}{dt} \, \delta s\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d^2 s}{dt^2} \, \delta s dt = 0.$$

Since δs is arbitrary for any intermediate time, it follows from this, first that for all times $\frac{d^2 s}{dt^2} = 0$, and so the velocity $\frac{ds}{dt} =$ const. (§ 71) and secondly that $\delta(s_1 - s_2) =$

0, where $s_1 - s_0$ denotes the length of the orbital curve. Thus the orbit is a geodetic line of the surface (§ 111).

§ 124. We shall next use the convenient form of Hamilton's principle to transform the equations of motion of a point-system from rectilinear right-angled co-ordinates to any arbitrary co-ordinates whatsoever. For in many cases it is found expedient to choose instead of rectilinear co-ordinates others which are better suited to the prescribed conditions of the system—for example, in the case of rotations the angle of rotation. We then need to take only so many co-ordinates as there are degrees of freedom in the system and we may regard these co-ordinates, which we shall call $q_1, q_2, \ldots$, as independent of one another. The rectilinear co-ordinates $x_1, y_1, z_1, x_2, \ldots$ are always definite functions of the q's which are known from the very beginning and which, if the prescribed conditions are dependent on the time, also contain the time t explicitly. If this is not the case the velocity components $\dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{x}_2$ are definite homogeneous linear functions of the $\dot{q}$'s, whose coefficients may, however, be dependent on the q's.

Since the variations $\delta x_1, \ldots$ are at any rate homogeneous linear functions of the δq's, the expression for the virtual external work A in (400) and (392) has the form :

$$A = \Phi_1 \delta q_1 + \Phi_2 \delta q_2 + \ldots \qquad (403)$$

where the quantities Φ are given by the external forces and are called the external "generalized force-components" corresponding to the general co-ordinates q.

This definition of force is the most general that can be given at all; it links up with the universal concept of work, that is, of potential, and extends its significance to every kind of change of state which can be characterized by a change of a variable q. It is worthy of note that the dimensions of the generalized force-component adapt themselves according to the dimensions of q. If, for example, q is an angle, Φ, according to (327), is an angular momentum.

Furthermore, with regard to the variation of Lagrange's function, L is a definite function, regarded as known, of the second degree in $\dot{q}_1, \dot{q}_2, \ldots$, whose coefficients depend on $q_1, q_2, \ldots$, and possibly on t. Accordingly, since the time t is not subjected to variation:

$$\delta L = \frac{\partial L}{\partial \dot{q}_1}\delta\dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2}\delta\dot{q}_2 + \ldots + \frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \ldots \quad (404)$$

Let us imagine the expressions (403) and (404) substituted in (400) and all the variations that occur reduced to terms of the independent variations $\delta q_1, \delta q_2, \ldots$. This is accomplished in the case of the quantities $\delta\dot{q} = \delta\frac{dq}{dt} = \frac{d\delta q}{dt}$ by means of integration by parts, according to the scheme:

$$\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}}\delta\dot{q}dt = -\int_{t_0}^{t_1}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q dt,$$

since at the limits of the integral the variations $\delta q \ldots$ vanish just like the $\delta x, \ldots$

After each term behind the integral sign of (400) has received one of the variations $\delta q_1, \delta q_2, \ldots$ as a factor, we must have, if (400) is to remain valid and since these variations are mutually independent:

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \Phi_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \Phi_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad \ldots \quad (405)$$

These are the so-called Lagrange equations of motion "of the second kind" as contrasted with those, (384), of the first kind.

§ 125. As an example we determine the equations of motion of a free point in *polar co-ordinates* r, θ, ϕ. The external work is:

$$A = R\delta r + \Theta\delta\theta + \Phi\delta\phi \quad \ldots \quad (405a)$$

where R, Θ, Φ denote the corresponding force-components.

If no potential energy is present Lagrange's function is equal to the vis viva K, and hence, in view of (92):

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \quad . \quad . \quad (405b)$$

and consequently the required equations of motion (405) are:

$$\left.\begin{aligned} &\frac{d}{dt}(m\dot{r}) - mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = R \\ &\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 = \Theta \\ &\frac{d}{dt}(mr^2\sin^2\theta\dot{\phi}) = \Phi \end{aligned}\right\} \quad . \quad . \quad (405c)$$

a simple result which could have been obtained directly from (55) and (92) only by laborious calculations.

In a corresponding way we obtain for the *cylindrical co-ordinates* ρ, ϕ, z by (159):

$$\left.\begin{aligned} &\frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\phi}^2 = P \\ &\frac{d}{dt}(m\rho^2\dot{\phi}) = \Phi \\ &\frac{d}{dt}(m\dot{z}) = Z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (405d)$$

§ 126. We shall now derive the principle of vis viva directly from Lagrange's equations of the second kind and shall therefore assume from now onwards that the time t is not contained explicitly in the expression for Lagrange's function L. If we multiply the equations (405) in turn by $\dot{q}_1$, $\dot{q}_2$, . . . and add, we get for the work performed by the external forces in the time dt:

$$\Sigma\Phi_1 dq_1 = A = \Sigma\left(\dot{q}_1 d\left(\frac{\partial L}{\partial\dot{q}_1}\right) - \frac{\partial L}{\partial q_1}dq_1\right) \quad . \quad (406)$$

If we compare the expression on the right-hand side of the equation with the complete differential:

$$dL = \Sigma\left(\frac{\partial L}{\partial\dot{q}_1}d\dot{q}_1 + \frac{\partial L}{\partial q_1}dq_1\right)$$

then we see that the two expressions when added together give:

$$A + dL = \Sigma\left(\dot{q}_1 d\left(\frac{\partial L}{\partial \dot{q}_1}\right) + \frac{\partial L}{\partial \dot{q}_1} d\dot{q}_1\right),$$

and this is the complete differential of:

$$\Sigma \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1}.$$

Accordingly, we get $A = dE$ or the equation of vis viva, if we set:

$$E = \Sigma \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} - L \quad . \quad . \quad . \quad . \quad (407)$$

The agreement of this equation with the definition of the kinetic potential given in (401) follows at once if E is replaced by $K + U$ and L by $K - U$, and if we reflect that U does not depend on $\dot{q}$. For then it follows that:

$$K = \frac{1}{2} \Sigma \dot{q}_1 \frac{\partial K}{\partial \dot{q}_1} \quad . \quad . \quad . \quad . \quad . \quad (407a)$$

a relationship which always holds, since K is a homogeneous function of the second degree in the $\dot{q}$'s.

The relationship, which here appears for the first time in (407), between the energy and Lagrange's function has a significance which is essentially more general than that expressed in (401). For the equation (407) retains a definite sense even in the case where the energy E cannot be divided into kinetic and potential energy, as, for example, in electrodynamic processes.

§ 127. As we have seen in § 75 and, more generally, in § 118 the principle of vis viva loses its validity if the prescribed conditions contain besides the co-ordinates also the time explicitly. But the equations of motion (405) also remain valid in this general case, as is seen from the way in which they were derived. In the case of the motion of a point on a straight line which is rotating with constant angular velocity ω—a case which we have discussed in § 75 and in § 117:

$$L = K = \frac{m}{2}(\dot{r}^2 + r^2\omega^2).$$

and so by (405):

$$\frac{d}{dt}(m\dot{r}) - mr\omega^2 = 0,$$

which is in agreement with (278*b*).

As a further example we discuss the vibrations of a pendulum moving in a vertical plane, its length l changing in some definite given way. Then l is a given function of the time, and we obtain, if we take the angle of displacement ϕ as the only independent co-ordinate, by § 70:

$$K = \frac{m}{2}(l^2\dot{\phi}^2 + \dot{l}^2),\ U = -\ mgl\cos\phi + \text{const.}$$

Consequently, by (401):

$$L = \frac{m}{2}(l^2\dot{\phi}^2 + l^2) + mgl\cos\phi + \text{const.}$$

and, by (405), we get as the equation of motion:

$$\frac{d}{dt}(ml^2\dot{\phi}) + mgl\sin\phi = 0$$

or:

$$2\dot{l}\dot{\phi} + l\ddot{\phi} + g\sin\phi = 0 \quad . \quad . \quad . \quad (408)$$

an equation which differs from the equation (244) of a common pendulum in having the term $2\dot{l}\dot{\phi}$.

We can choose the rhythm with which the length alters so that the energy of the vibration is influenced predominantly in a definite sense. It is owing to the same circumstance that we may swing ourselves as high as we wish by properly timed impulses.

§ 128. The first quantities that occur in Lagrange's equations (405):

$$\frac{\partial L}{\partial \dot{q}_1} = p_1,\ \frac{\partial L}{\partial \dot{q}_2} = p_2,\ . \ . \ . \quad . \quad . \quad (409)$$

are called the *generalized momenta* corresponding to the general co-ordinates q. They are linear homogeneous functions of the velocities $\dot{q}$—that is, they may be expressed simply in terms of the $\dot{q}$'s and conversely. Corresponding to the rectilinear co-ordinates x, y, z we have the momenta

$m\dot{x}, m\dot{y}, m\dot{z}$, which are obtained directly from the expression for L.

It is often found useful to characterize the state of the point-system by using, in addition to the co-ordinates q, the momenta p in place of the velocities $\dot{q}$. The equations of motion then assume a particularly simple form if the energy E is introduced as the characteristic function instead of the Lagrange function L. We then write the equation (406) for $A = dE$ in the form :

$$dE = \Sigma\left(\dot{q}_1 dp_1 - \frac{\partial L}{\partial q_1} dq_1\right)$$

and consider E as a function of the q's and p's ;* from this it directly follows that :

$$\frac{\partial E}{\partial p_1} = \dot{q}_1, \frac{\partial E}{\partial p_2} = \dot{q}_2, \ldots \ldots \quad (410)$$

and :

$$\left(\frac{\partial E}{\partial q_1}\right)_p = -\left(\frac{\partial L}{\partial q_1}\right)_{\dot{q}}, \left(\frac{\partial E}{\partial q_2}\right)_p = -\left(\frac{\partial L}{\partial q_2}\right)_{\dot{q}}, \ldots \ldots \quad (411)$$

The suffixes outside the brackets are to denote that E is to be differentiated while p is kept constant and L is to be differentiated while $\dot{q}$ is kept constant.

By using the last relationships we may write the equations (405) as follows :

$$\left.\begin{array}{l} \dfrac{dq_1}{dt} = \dfrac{\partial E}{\partial p_1}, \dfrac{dp_1}{dt} = -\dfrac{\partial E}{\partial q_1} + \Phi_1 \\ \dfrac{dq_2}{dt} = \dfrac{\partial E}{\partial p_2}, \dfrac{dp_2}{dt} = -\dfrac{\partial E}{\partial q_2} + \Phi_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad (412)$$

These are "Hamilton's canonical equations of motion." From them we obtain, if besides the external forces also the energy E is given as a function of the q's and p's, all the q's and p's as functions of the time t and of those constants which refer to the initial state.

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* Here the q's and p's are no longer assumed to be functions of the one variable t but independent variables. This is justified by the circumstance that the equation is valid for all arbitrary forces, and hence also for any arbitrary change of state of the point-system.

Also the energy principle follows directly from the equations (412), if we use it to form the expression for the complete differential dE.

If the external forces Φ are given as functions of the time t the equations of motion (412) may be simplified in form by introducing " Hamilton's function " :

$$H = E - \Sigma\Phi_1 q_1 \quad . \quad . \quad . \quad (412a)$$

The equations (412) then become :

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \; \frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}, \; . \; . \; . \quad . \quad . \quad (412b)$$

For a closed system ($\Phi = 0$) Hamilton's function H becomes coincident with the energy E, and we obtain :

$$\frac{dq_1}{dt} = \frac{\partial E}{\partial p_1}, \; \frac{dp_1}{dt} = -\frac{\partial E}{\partial q_1}, \; . \; . \; . \quad . \quad . \quad (413)$$

§ 128*a*. A general method of integrating the equations of motion (413) that hold for a *closed system* may be derived by considering a little more closely the " action integral " or the " action function " :

$$W = \int_{t_0}^{t_1} L dt \quad . \quad . \quad (414)$$

which occurs in Hamilton's principle.

This quantity has a perfectly definite value for the real motion, if the initial and final positions of the system are given. By (400) this value is characterized by:

$$\delta W = \int_{t_0}^{t_1} \delta L \, . \, dt = 0 \quad . \quad . \quad . \quad (415)$$

for every variation of the motion.

Let us next inquire what value δW assumes if we also vary the initial and the final positions of the system—that is, the initial and the final values of the co-ordinates $q_1, q_2, \ldots$, while the time t remains unvaried throughout. The answer to this question is obtained by calculating

δW on the basis of equation (404), exactly on the lines there followed, by means of integration by parts; only the circumstance need be taken into account that the variations δq_1, δq_2, . . . do not now vanish at the limits of the integral. We thus obtain, if we use the equations of motion (405) for the closed system ($\Phi = 0$):

$$\delta W = \Sigma\left[\frac{\partial L}{\partial \dot{q}_1}\delta q_1\right]_{t_0}^{t_1} = \Sigma\left[p_1 \delta q_1\right]_{t_0}^{t_1} \quad . \quad . \quad (416)$$

According to our above remarks, we may regard W as a definite function of the initial co-ordinates, the final co-ordinates and the times t_0 and t_1. If from now on we call the initial co-ordinates $q_1{}^0$, $q_2{}^0$, . . . and the final co-ordinates q_1, q_2, . . . , then, since the time is not subjected to variation:

$$\delta W = \Sigma \frac{\partial W}{\partial q_1{}^0}\delta q_1{}^0 + \Sigma \frac{\partial W}{\partial q_1}\delta q_1$$

and comparison with (416) gives:

$$\frac{\partial W}{\partial q_1} = p_1, \quad \frac{\partial W}{\partial q_2} = p_2, \ldots \quad . \quad . \quad . \quad (417)$$

$$\frac{\partial W}{\partial q_1{}^0} = -p_1{}^0, \quad \frac{\partial W}{\partial q_2{}^0} = -p_2{}^0, \ldots \quad . \quad (418)$$

We shall now also take the dependence of the integral of action W on the time t_1 into consideration; and henceforward we shall denote the time t_1 by t for brevity. From (414) we have for this dependence:

$$\frac{dW}{dt} = L \quad . \quad . \quad . \quad . \quad . \quad (419)$$

where the differentiation of W is to be " total "—that is, such that the co-ordinates q_1, q_2, . . . change with the time t in accordance with the actual motion, while the initial co-ordinates $q_1{}^0$, $q_2{}^0$, . . . as well as t_0 are kept constant. That is, we have:

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial q_1}\cdot\dot{q}_1 + \frac{\partial W}{\partial q_2}\cdot\dot{q}_2 + \ldots,$$

where $\frac{\partial W}{\partial t}$ now refers to "partial" differentiation, the co-ordinates being kept constant. If we take into account (417) and (419), the last equation gives :

$$\frac{\partial W}{\partial t} + p_1\dot{q}_1 + p_2\dot{q}_2 + \ . \ . \ . \ - L = 0,$$

or, by (409) and (407) :

$$\frac{\partial W}{\partial t} + E = 0.$$

If we here regard the energy, as in (413), as a known function of the p's and q's, which we shall indicate by means of the term $E_{p,\,q}$, and replace the momentum co-ordinates p in it by their values in (417), the last relationship runs :

$$\frac{\partial W}{\partial t} + E_{q,\,\frac{\partial W}{\partial q}} = 0 \quad . \quad . \quad . \quad . \quad . \qquad (420)$$

It shows that the integral of action W, regarded as a function of $q_1, q_2, \ . \ . \ . \ q_1^0, q_2^0, \ . \ . \ . \ t, t_0$ satisfies a definite partial differential equation—the Hamilton-Jacobi differential equation.

§ 128*b*. Exactly as we can prove the validity of the differential equation (420) from the expression for the action integral W for a definite motion, so conversely by integrating (420) we can find a function W which represents the action integral of a motion, and closer inspection shows that *every* integral W of the differential equation, which contains besides the variables $q_1, q_2, \ . \ . \ . \ t$ and the additive integration constants just as many constants of integration $\alpha_1, \alpha_2, \ . \ . \ .$ as the system in question has degrees of freedom, gives the *general* integral of the equations of motion (413). If by (417) we write, assuming that such a function W has been found :

$$\frac{\partial W}{\partial q_1} = p_1, \ \frac{\partial W}{\partial q_2} = p_2, \ . \ . \ . \ . \qquad . \qquad (421)$$

and remembering that, by (418) :

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \quad \frac{\partial W}{\partial \alpha_2} = \beta_2, \ldots \qquad (422)$$

then we have altogether for n degrees of freedom $2n$ equations, which may serve to calculate the $2n$ variables $q_1, q_2, \ldots p_1, p_2, \ldots$ as functions of the time t and the $2n$ integration constants $\alpha_1, \alpha_2 \ldots, \beta_1, \beta_2, \ldots$ of the equations of motion (413).

The fact that when (420), (421) and (422) are valid the equations of motion (413) are also satisfied may easily be shown in the following way. If we differentiate the first equation (422) "totally" with respect to t, we get :

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \alpha_1}\right) = 0,$$

or :

$$\frac{\partial^2 W}{\partial \alpha_1 \partial t} + \sum_i \frac{\partial^2 W}{\partial \alpha_1 \partial q_i} \frac{dq_i}{dt} = 0 \quad . \qquad (423)$$

where the summation is to be extended from 1 to n. In the same way, $n - 1$ further equations follow, which together with (423) enable us to calculate the n velocities $\dot{q}$.

On the other hand, we get by differentiating (420) with respect to α_1, using (421) :

$$\frac{\partial^2 W}{\partial \alpha_1 \partial t} + \sum_i \frac{\partial E}{\partial p_i} \frac{\partial^2 W}{\partial \alpha_1 \partial q_i} = 0 \quad . \qquad (424)$$

and, in a corresponding way, the $n - 1$ other equations, so that from these n equations we can calculate the n quantities $\frac{\partial E}{\partial p}$ uniquely. But the coefficients in the systems of equations (423) and (424), the second partial differential coefficients of W, are absolutely identical. Hence it follows that the n roots of the equations are also identical—that is, that for every i :

$$\frac{dq_i}{dt} = \frac{\partial E}{\partial p_i} \quad . \quad . \quad . \quad . \quad . \qquad (425)$$

and so the first half of the equations of motion (413) is satisfied.

Concerning the second half, it follows by differentiating (420) with respect to q_1 that :

$$\frac{\partial^2 W}{\partial q_1 \partial t} + \frac{\partial E}{\partial q_1} + \sum_i \frac{\partial E}{\partial p_i} \frac{\partial^2 W}{\partial q_i \partial q_1} = 0$$

or, by (425) :

$$\frac{\partial^2 W}{\partial q_1 \partial t} + \sum_i \frac{\partial^2 W}{\partial q_1 \partial q_i} \frac{dq_i}{dt} + \frac{\partial E}{\partial q_1} = 0$$

or finally :

$$\frac{d}{dt}\left(\frac{\partial W}{\partial q_1}\right) + \frac{\partial E}{\partial q_1} = 0,$$

from which, by (421), the validity of the other half of the equations is established.

The equations (422), although related in form to the equations (418), are essentially more general, because the constants α need not be the initial values of the co-ordinates q.

§ 128*c*. Since in the Hamilton–Jacobi differential equation (420) the time occurs only as a differential, it admits of an integral of the form :

$$W = -\alpha_1 t + V \quad . \quad . \quad . \quad . \quad (426)$$

where V depends only on the co-ordinates $q_1, q_2, \ldots$ and the constants $\alpha_1, \alpha_2, \ldots .$ We then obtain for the function V the condition :

$$E_{q, \frac{\partial V}{\partial q}} - \alpha_1 = 0 \quad . \quad . \quad . \quad . \quad (427)$$

Here the constant α_1 represents the total energy of the system. The equations (421) and (422) then become :

$$\frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \ldots \quad . \quad . \quad (428)$$

and :

$$\frac{\partial V}{\partial \alpha_1} = t + \beta_1, \quad \frac{\partial V}{\partial \alpha_2} = \beta_2, \ldots \quad . \quad . \quad (429)$$

The integration of (427) can sometimes be successfully effected by separating the variables—namely, when the left-hand side of the equation may be represented as a function of n arguments, each of which depends only on a single co-ordinate q_i and the corresponding differential coefficient $\frac{\partial V}{\partial q_i}$. Then we may set each individual argument equal to a constant α. Further, we may assume:

$$V = V_1 + V_2 + V_3 + \ldots + V_n \quad . \quad . \quad (430)$$

where each of the quantities V_1, V_2, . . . depends only on a single co-ordinate q_1, q_2, . . ., thus:

$$\frac{\partial V}{\partial q_1} = \frac{\partial V_1}{\partial q_1}, \frac{\partial V}{\partial q_2} = \frac{\partial V_2}{\partial q_2}, \ldots \quad . \quad . \quad (431)$$

and so we obtain for each of these functions a special differential equation which may be solved by direct integration.

§ 128*d*. As a simple example of the application of the Hamilton–Jacobi differential equation we consider the planetary motion already treated in § 52 *et seq.*, and we shall use the same notation as previously. We then have, for two degrees of freedom:

$$q_1 = r, \; q_2 = \phi,$$

and further the energy as a function of the co-ordinates and velocities:

$$E = K + U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{fm\mu}{r} \quad . \quad (432)$$

the kinetic potential:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{fm\mu}{r} \quad . \quad . \quad . \quad (433)$$

the momentum co-ordinates (generalized momenta):

$$p_1 = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \; p_2 = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad . \quad . \quad (434)$$

and the energy as a function of the co-ordinates and momenta:

$$E = \frac{1}{2m}{p_1}^2 + \frac{1}{2mr^2}{p_2}^2 - \frac{fm\mu}{r} \quad . \quad . \quad (435)$$

Hence the differential equation (427) here runs :

$$\frac{1}{2m}\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial V}{\partial \phi}\right)^2 - \frac{fm\mu}{r} = \alpha_1 . \qquad (436)$$

This equation may be integrated by using (430) and writing :

$$\left.\begin{aligned} V &= V_1 + V_2 \\ \frac{\partial V}{\partial \phi} &= \frac{\partial V_2}{\partial \phi} = \alpha_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (437)$$

and consequently :

$$\frac{\partial V}{\partial r} = \frac{\partial V_1}{\partial r} = \sqrt{2m\alpha_1 + \frac{2fm^2\mu}{r} - \frac{\alpha_2{}^2}{r^2}}$$

We then get, if we omit the unimportant additive constants :

$$\left.\begin{aligned} V_2 &= \alpha_2\phi \\ V_1 &= \int \sqrt{2m\alpha_1 r^2 + 2fm^2\mu r - \alpha_2{}^2}\cdot\frac{dr}{r} \end{aligned}\right\} \quad . \qquad (438)$$

and from (428), (429), (434) and (437) :

$$p_1 = m\dot{r} = \frac{\partial V_1}{\partial r} = \sqrt{2m\alpha_1 + \frac{2fm^2\mu}{r} - \frac{\alpha_2{}^2}{r^2}} \quad . \qquad (439)$$

$$p_2 = mr^2\dot{\phi} = \frac{\partial V_2}{\partial \phi} = \alpha_2 \quad . \quad . \qquad (440)$$

$$t + \beta_1 = \frac{\partial V_1}{\partial \alpha_1} = \int \frac{mr\,.\,dr}{\sqrt{2m\alpha_1 r^2 + 2fm^2\mu r - \alpha_2{}^2}} \quad . \qquad (441)$$

$$\begin{aligned} \beta_2 = \frac{\partial V_1}{\partial \alpha_2} + \phi &= \phi - \int \frac{\alpha_2\cdot dr}{r\sqrt{2m\alpha_1 r^2 + 2fm^2\mu r - \alpha_2{}^2}} \\ &= \phi - \cos^{-1}\frac{\alpha_2{}^2 - fm^2\mu r}{r\sqrt{f^2m^4\mu^2 + 2m\alpha_1\alpha_2{}^2}} \quad . \qquad (442) \end{aligned}$$

These four equations show themselves to be fully identical in form with the equations of § 53, if we set :

$$\alpha_1 = \frac{mc}{2},\ \alpha_2 = mc',\ \beta_2 = c''.$$

§ 129. In nature neither the expression for the energy nor that for Lagrange's function is directly given. Hence in applying the theory it is of the greatest importance that there is yet another theorem of very general character, which also holds : *the principle of the equality of action and reaction.* We have hitherto introduced and used this principle only for point-systems at rest. Moreover, the argument by which we made its validity in nature plausible (§ 29) cannot be applied to points which are moving arbitrarily and whose distances vary with the time, particularly when non-conservative forces, such as friction, come into consideration.

Hence it is doubly important to convince ourselves that the principle of action and reaction is intimately connected with the universal principle of the conservation of energy; this is accomplished by means of the principle of relativity (§ 59).

For let us imagine two material points 1 and 2, between which some forces or other act, moving in an arbitrary way. Then by (147) the change in the sum of the amounts of vis viva which occur in the time *dt* is :

$$\begin{aligned} dK = {} & X_1dx_1 + Y_1dy_1 + Z_1dz_1 \\ & + X_2dx_2 + Y_2dy_2 + Z_2dz_2 \quad . \quad . \quad (443) \end{aligned}$$

The principle of relativity states that this quantity dK remains unchanged if we pass from the stationary co-ordinate system to a co-ordinate system in uniform motion by means of the equations (194), no matter whether the forces have a potential or not. For the principle of relativity holds not only for mechanical, but for *all* physical events—for example, also for the transformation of mechanical energy into heat; that is, the amount of mechanical energy transformed into heat is independent of any uniform rectilinear motion of the co-ordinate system.

Hence we get the following relationship for the transformation of co-ordinates above mentioned, by (443) :

$$X_1dx_1 + \ldots + X_2dx_2 + \ldots = X'_1dx'_1 + \ldots + X'_2dx'_2 + \ldots$$

and in view of (192) and (194):

$$X_1 dx_1 + \ldots + X_2 dx_2 + \ldots = X_1(dx_1 - u_0 dt) + \ldots \\ + X_2(dx_2 - u_0 dt) + \ldots$$

or:

$$(X_1 + X_2)u_0 + (Y_1 + Y_2)v_0 + (Z_1 + Z_2)w_0 = 0,$$

a relationship which is obeyed for any values of u_0, v_0, w_0, if we have quite generally:

$$X_1 = -X_2,\ Y_1 = -Y_2,\ Z_1 = -Z_2 \quad . \quad (444)$$

in agreement with the principle of the equality of action and reaction.

§ 130. The full meaning and fruitfulness of the principle last mentioned come into evidence if we again, as before in statics (§ 112), divide all the forces that are active in a system of material points into *internal* and *external* forces. If we then also adduce d'Alembert's principle (§ 66) we may immediately enunciate the following theorem:

In every motion of a system of material points the external forces and the inertial resistances are in equilibrium in the system supposed rigid.

To clothe this fundamental law in analytical language we denote the external forces (driving forces or forces of constraint) by $\boldsymbol{F}_a$. Then, in accordance with the conditions of equilibrium (306) of a rigid body:

$$\Sigma(\boldsymbol{F}_a - m\ddot{\boldsymbol{r}}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (445)$$

$$\Sigma[\boldsymbol{r}, (\boldsymbol{F}_a - m\ddot{\boldsymbol{r}})] = 0 \quad . \quad . \quad . \quad . \quad (446)$$

These six equations form the common starting-point for the whole mechanics of rigid, solid, liquid and gaseous bodies. We shall elucidate their meaning by describing a few general applications.

§ 131. Equations (445) assume a very simple form if we introduce the position of the centre of gravity by (287).

Accordingly we have by differentiating twice with respect to the time t:

$$\ddot{\boldsymbol{r}}_0 \Sigma m = \Sigma m\ddot{\boldsymbol{r}} \quad . \quad . \quad . \quad . \quad . \quad (447)$$

so that equation (445) runs:

$$\ddot{\mathbf{r}}_0 \Sigma m = \Sigma \mathbf{F}_a \quad . \quad . \quad . \quad . \quad . \qquad (448)$$

That is, the centre of gravity of a system of material points moves as if the whole mass of the system were concentrated in it and as if all the external forces acted on it. Hence the internal forces play no part at all in the motion of the centre of gravity.

For example, if we fling any solid or liquid body freely into the air, its centre of gravity moves in the parabola which is prescribed by its initial conditions, so long as no external force except the weight of the individual parts of the body come into question.

Even an explosion of the body cannot disturb this parabolic path, so long as none of the pieces strikes an external obstacle. In the same way, the explosion of a planet would not prevent its centre of gravity from continuing its elliptic motion about the sun.

Only an external force is able to impart an acceleration to the centre of gravity.

Even the strongest man, if placed on an absolutely smooth surface, is unable to move from his position if he is initially at rest, or to stop if he is in motion. From this we see the great importance of the friction of the earth's surface, of street pavements, of railway lines, for moving heavy loads. When a horse draws a wagon, the force of the traces acts with the same intensity (and *not* more) on the wagon forwards as on the horse backwards, but for the wagon this is the only force that comes into question, whereas for the horse the friction also acts, which his hoofs experience when pushed against the ground and which must act in the forward direction and be at least great enough to overcome the tension of the traces.

Moreover the question which we proposed earlier in § 82 and left unanswered—namely, regarding the way in which a resting rigid body is set into motion by a couple—receives the first part of its answer here. For by equation (448) the centre of gravity of the body remains at rest.

Thus the motion is a rotation about the centre of gravity. But we must not suppose that the axis of rotation always coincides with the axis of the couple. We shall discuss this relationship later in § 149.

§ 132. If no external forces act at all or if the external forces satisfy the equation $\Sigma \boldsymbol{F}_a = 0$, then by (448) $\ddot{\boldsymbol{r}} = 0$—that is, the centre of gravity moves with uniform motion in a straight line (conservation of the motion of the centre of gravity).

We may also formulate this theorem in another way by introducing the momenta (§ 128); we write the integral of $\Sigma m\ddot{\boldsymbol{r}} = 0$ in the form :

$$\Sigma m\dot{\boldsymbol{r}} = \Sigma m\boldsymbol{q} = \text{const.} \quad . \quad . \quad . \quad (449)$$

We call this vector, which is constant in magnitude and direction, the " resultant momentum " of the point-system. This is compounded from the momenta of the individual points just like the resultant force is compounded from the individual forces. The equation (449) then expresses the law of the conservation of momentum or impulse. A characteristic difference between this law and that of the conservation of energy consists in the circumstance that momentum is a vector but energy is a scalar; that is why the conservation of momentum is expressed by three equations in contrast with the one equation of the conservation of energy.

An example of the law of conservation of momentum is given by the recoil associated with the firing of a cannon, in which process the projectile and the cannon move apart with momenta which are equal and opposite, since the resultant momentum was zero originally and must remain zero so long as no external force comes into play.

§ 133. As a further example of the law of conservation of momentum we shall discuss the laws governing the *collision* of two material points 1 and 2 moving on the same straight line—say, the x-axis. Let their co-ordinates be x_1 and $x_2 > x_1$, and their velocities before the collision

u_1 and u_2. In order that a collision may occur at all we must have:

$$u_1 > u_2 \quad . \quad . \quad . \qquad (450)$$

where the quantities u may be positive or negative.

A collision is a very complicated process, in which intense forces come into action; these, however, last for only a short time and depend to a high degree on the material constitution of the points. This makes it the more important to derive from general mechanics relationships of wide application for the magnitude of the velocities after the collision.

In the absence of every external force the law of conservation of momentum holds for the system of two material points:

$$m_1u_1 + m_2u_2 = m_1u_1' + m_2u_2' \quad . \quad . \qquad (451)$$

where u_1' and u_2' denote the velocities after the collision. To determine u_1' and u_2' uniquely this equation is not, of course, sufficient. For this purpose we require a further condition which will be different according to the material constitution of the points. But the following inequality must be obeyed in all circumstances:

$$u_1' \leqq u_2' \quad . \quad . \quad . \quad . \quad . \qquad (452)$$

since the points are assumed to be impenetrable.

Among all the possible cases we consider those two which are of particular interest as ideal limiting cases.

§ 134. **Inelastic Collision**. A collision is said to be perfectly inelastic when the two points do not rebound from each other but continue their path as one body. Hence we have for this case:

$$u_1' = u_2' = u' \quad . \quad . \quad . \qquad (453)$$

and by (451):

$$u' = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \quad . \quad . \quad . \quad . \qquad (454)$$

If the masses are equal, the velocity after the collision is the arithmetic mean of the velocities before the collision. If the momenta are equal and opposite, the common final velocity is zero.

In an inelastic collision mechanical energy is always lost. Let us calculate the amount of this loss. The difference in the values of the vis viva of the two points before and after the collision is :

$$\frac{1}{2}(m_1 u_1^2 + m_2 u_2^2 - (m_1 + m_2) u'^2)$$

or by (454) :

$$\frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2 \quad . \quad . \quad . \quad (455)$$

which is positive.

According to the universal energy principle, a corresponding amount of molecular energy must be gained in the form of heat, deformation or electricity.

The fact that this transformation of energy always occurs in one direction, in that of a decrease of the vis viva of the motion, just as in the case of friction, already points to the controlling power of some universal law, which is in itself foreign to the energy-principle; this law has received exact expression in the second law of thermodynamics.

§ 135. **Elastic Collision.** A collision is called perfectly elastic when the two material points undergo only transitory and not lasting changes in their molecular constitution—that is, are neither warmed, nor permanently deformed, nor changed in their molecular energy.

By the principle of conservation of energy the sum of the amounts of vis viva of the points, being the only kind of energy which comes into consideration here, has the same value after as before the collision, thus :

$$\frac{1}{2}(m_1 u_1^2 + m_2 u_2^2) = \frac{1}{2}(m_1 u_1'^2 + m_2 u_2'^2) \quad . \quad (456)$$

or :

$$m_1 (u_1^2 - u_1'^2) = m_2 (u_2'^2 - u_2^2),$$

and by dividing by the equation (451), after it has been suitably transposed:

$$u_1 + u_1' = u_2 + u_2' \quad . \quad . \quad . \quad (457)$$

from which we obtain, in conjunction with (451), the values:

$$\left.\begin{aligned} u_1' &= \frac{(m_1 - m_2)\,u_1 + 2m_2u_2}{m_1 + m_2} \\ u_2' &= \frac{2m_1u_1 - (m_1 - m_2)\,u_2}{m_1 + m_2} \end{aligned}\right\} \quad . \quad . \quad (458)$$

which satisfy the inequality (452) if we take (450) into account.

If the masses are equal, then we have $u_1' = u_2$ and $u_2' = u_1$. That is, the velocities simply exchange values. If m_2 is very great compared with m_1 ($m_2 >> m_1$), then we get, if we divide the numerator and denominator by m_2:

$$\left.\begin{aligned} u_1' &= -\,u_1 + 2u_2 \\ u_2' &= u_2 \end{aligned}\right\} \quad . \quad . \quad . \quad (459)$$

This case corresponds with the elastic reflection of a material point, moving with a velocity u_1 which strikes a wall moving with the velocity u_2. If $u_2 = 0$, the point rebounds from the stationary wall with the same velocity u_1; if $u_2 = \frac{u_1}{2}$, the point comes to rest permanently; if $u_2 > \frac{u_1}{2}$ the point follows the wall at a gradually increasing distance. The fact that the principle of conservation of energy is preserved, although the velocity of the point changes while that of the wall remains essentially unaltered, is guaranteed by equation (456).

§ 136. The process involved in a collision has given rise to the introduction of the concept of *instantaneous forces*. These are forces which differ from zero only during an extremely short time τ, but during this time are of such magnitude that they produce an appreciable change of velocity. To find the characteristic features of an instantaneous force we assume in the general equations (412) the external components of force $\Phi_1, \Phi_2, \ldots$ as instantaneous forces, which act from the time t to the time $t + \tau$, and integrate the equations which refer to the co-ordinates $q_1, q_2, \ldots$ with respect to t from t to $t + \tau$. We then obtain, if we denote the values referred to the moment of time $t + \tau$ by a dash :

$$q_1' - q_1 = 0,\ p_1' - p_1 = \int_t^{t+\tau} \Phi_1 dt, \text{ and so forth} \quad . \quad (460)$$

For on account of the smallness of τ the time-integrals, which are to be taken over finite quantities, vanish. Hence, whereas the co-ordinates q themselves are not appreciably changed by the instantaneous forces, since the velocities always remain finite, the momenta (and with them the velocities) undergo a sudden change, whose value is represented by the instantaneous force in question. Thus this time-integral is characteristic of the action of the instantaneous force; it is called the " impulse " $i_1, i_2, \ldots$ of the force.

If, as described in § 128, we use instead of the velocities $\dot{q}$ the momenta p to characterize the state of a point-system in addition to the co-ordinates q, we may imagine these momenta to be produced—in a way that is easily pictured—by instantaneous forces, whose impulses are equal to the quantities p. Hence the momenta p are also called *impulse co-ordinates*.

The principle of action and reaction holds for impulses, as for all other kinds of force—that is, corresponding to every impulse which is exerted on one material point by another there is an equally great and opposite impulse exerted by the second point on the first.

Let us now also calculate the work A done by these instantaneous forces. This work is equal to the change of energy E or, what comes to the same thing, to the vis viva K, because, by (460), the potential energy is not appreciably changed by the instantaneous forces. Thus :

$$A = K' - K,$$

and by (407*a*) and (409):

$$A = \frac{1}{2}\Sigma\dot{q}_1{}'p_1{}' - \frac{1}{2}\Sigma\dot{q}_1 p_1.$$

Since K is a complete homogeneous quadratic function of the $\dot{q}$'s (and of the p's), we have by (409):

$$\Sigma\dot{q}_1{}'p_1 = \Sigma\dot{q}_1 p_1{}' \quad . \quad . \quad . \quad . \quad (461)$$

Consequently we may also write:

$$A = \frac{1}{2}\Sigma(\dot{q}_1{}' + \dot{q}_1)\cdot(p_1{}' - p_1)$$

or:

$$A = \Sigma\frac{\dot{q}_1{}' + \dot{q}_1}{2}\cdot \boldsymbol{i}_1 \quad . \quad . \quad . \quad . \quad (462)$$

That is, the work of the instantaneous forces is obtained by multiplying the impulses by the arithmetic mean of the velocities before and after the collision.

The advantage of introducing instantaneous forces is shown, for example, in deriving the laws of elastic collision treated in § 135. For this we have by (462):

$$\frac{\boldsymbol{u}_1{}' + \boldsymbol{u}_1}{2}\boldsymbol{i}_1 + \frac{\boldsymbol{u}_2{}' + \boldsymbol{u}_2}{2}\boldsymbol{i}_2 = 0.$$

On the other hand we have by the principle of action and reaction:

$$\boldsymbol{i}_1 + \boldsymbol{i}_2 = 0,$$

and from this equation (457) follows in a simpler way than before.

§ 137. We shall now consider the general equation (446) a little more closely, and for this purpose we write it in the form:

$$\Sigma[\boldsymbol{r}, m\dot{\boldsymbol{r}}] = \Sigma[\boldsymbol{r}, \boldsymbol{F}_a] \quad . \quad . \quad . \quad . \quad (463)$$

Let us next take the special case where the right-hand side of the equation vanishes, as when no external forces at all are acting, thus:

$$\Sigma[\boldsymbol{r}, m\dot{\boldsymbol{r}}] = 0 \quad . \quad . \quad . \quad . \quad . \quad (464)$$

Each individual term of this sum is called the "moment of momentum" or the "impulse-moment" (*Drehimpuls* or *Impulsmoment*) of the point-mass in question with respect to the origin of co-ordinates; and the whole sum is called the " resultant moment " of all the momenta, or the " resultant moment of momentum " with respect to this point. The notation, the law of formation and the compounding of moments of momentum correspond exactly, according to §§ 85 to 88, with the laws which hold for the moments of forces. Similarly, the direction of the vector (464) is called the " axis " of the resultant moment of momentum.

Equation (464) acquires a concrete kinematic meaning if we reflect that according to the discussion in § 50 the projection of the resultant moment of momentum on to any plane described through the origin of co-ordinates—that is, the component of the vector (464) in the direction of the normal to this plane—is equal to the algebraic sum of twice the " areal velocity " multiplied by the mass of each individual point; these areal velocities are measured by the surfaces described by the radius vectors of the points in the plane in question, being taken as positive or negative according to the sense of the individual rotations. If the plane selected passes through the axis of the resultant moment of momentum, the algebraic sum of twice the areal velocities multiplied by the respective masses becomes equal to zero, because the component of a vector in a direction perpendicular to its own direction vanishes. But if the plane is perpendicular to the axis, this sum is a maximum, being equal to the absolute value of the resultant moment of momentum (464). This favoured plane which remains fixed in space for all times is therefore called the " invariable plane " and the theorem expressed in (464) is called the " principle of sectorial areas."

If the constant (464) is equal to zero, the resultant moment of momentum vanishes at all times. But this does not allow us, as in the case of the momentum in § 132, to derive a law of conservation, because the equation cannot in general be integrated with respect to the time. Hence, for example, a person who stands freely on an absolutely smooth surface cannot impart to himself a velocity of rotation, but he can effect a turn, say by stretching his right arm out sideways, then passing it round in a horizontal semicircle in front, to the left and finally drawing it to his side again. This movement, if repeated sufficiently often, effects a rotation of the person about an angle of any arbitrary size to the right. This explanation also accounts for the famous apparent paradox of the falling cat, which always manages to drop on to its feet.

§ 138. In general, it is not possible to integrate equation (463), but it may often be simplified by introducing in place of the stationary co-ordinate system a moving system whose axes remain parallel to themselves and whose origin is at the centre of gravity, in accordance with the equations (191), which run, in vectorial form :

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{r}_0,\ \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}' + \dot{\boldsymbol{r}}_0,\ \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}' + \ddot{\boldsymbol{r}}_0. \quad . \quad (465)$$

where, on account of (287) :

$$\Sigma m\boldsymbol{r}' = 0,\ \Sigma m\dot{\boldsymbol{r}}' = 0,\ \Sigma m\boldsymbol{r}' = 0 \quad . \quad . \quad (466)$$

Since, further, by (192) $\boldsymbol{F}_a = \boldsymbol{F}_a'$, we get if we substitute accented co-ordinates in (463) :

$$\Sigma[\boldsymbol{r}' + \boldsymbol{r}_0,\ m(\ddot{\boldsymbol{r}}' + \ddot{\boldsymbol{r}}_0)] = \Sigma[\boldsymbol{r}' + \boldsymbol{r}_0,\ \boldsymbol{F}_a']$$

or :

$$\Sigma[\boldsymbol{r}',\ m\ddot{\boldsymbol{r}}'] + \Sigma[\boldsymbol{r}_0,\ m\ddot{\boldsymbol{r}}'] + \Sigma[\boldsymbol{r}',\ m\ddot{\boldsymbol{r}}_0] + \Sigma[\boldsymbol{r}_0,\ m\ddot{\boldsymbol{r}}_0]$$
$$= \Sigma[\boldsymbol{r}',\ \boldsymbol{F}_a'] + \Sigma[\boldsymbol{r}_0,\ \boldsymbol{F}_a'].$$

In this equation the second and third terms on the left-hand side vanish on account of (466), as we see by calculating any component; but the fourth term is, by (448), equal to the second term on the right-hand side of the equation, so that we are finally left with :

$$\Sigma[\boldsymbol{r}',\ m\ddot{\boldsymbol{r}}'] = \Sigma[\boldsymbol{r}',\ \boldsymbol{F}_a'] \quad . \quad . \quad . \quad (467)$$

and this is again precisely the equation (463), but now with accented quantities. Its particular importance consists in the fact that the resultant moment of the external forces with respect to the centre of gravity often has a simpler value than that with respect to an origin which is fixed in space. If this moment is equal to zero, as, for example, in the case of gravitation, then the principle of sectorial areas (464) holds for the relative motion of the point-system about the centre of gravity, no matter in how complicated a way the centre of gravity may move. If we throw a heavy rigid body into the air in any way, it turns, if we disregard the resistance of the air, in exactly the same way about its centre of gravity as if the centre of gravity were at rest and no external force were acting at all. This theorem combined with the theorem of § 131 on the motion of the centre of gravity enables us to answer fully the question as to how the body moves. A similar inference may be drawn with regard to the rotation of a planet about its centre of gravity.

CHAPTER IV

DYNAMICS OF A RIGID BODY

§ 139. We shall now apply the general equations of dynamics to the motion of a rigid body and shall first convince ourselves that in every case the six equations (445) and (446) suffice to solve the problem completely. For if the body is completely free it has, according to § 103, six degrees of freedom, corresponding to the six equations of motion, in which all the external forces $\boldsymbol{F}_a$ are to be regarded as given. But if the motion of the body is restricted from the outset by prescribed conditions, the external forces partly consist of forces of constraint, and the equations which contain these forces of constraint cannot serve to determine the motion. But we already know from § 91 that in the case of equilibrium the driving forces alone must always satisfy just as many equations as there are degrees of freedom; and it is just these equations, generalized by the addition of inertial resistances, which also contain the laws of the motion. If the motion of the body has been determined in this way we can derive from the remaining equations the constraining forces which are necessary to maintain the prescribed conditions during this motion.

§ 140. Let us first take a body which can be rotated about a *fixed axis;* it has one degree of freedom. We take the axis of rotation as the z-axis. We suppose the driving forces to be given and to be compounded, according to § 88, into a single resultant $\boldsymbol{F}$ that acts at the origin, and a couple $\boldsymbol{N}$; in the same way, we shall suppose the forces of constraint, which are unknown at the outset, to be compounded into a resultant $\boldsymbol{F}'$ and the couple $\boldsymbol{N}'$.

Then of the six equations (445) and (446) only the last contains no member which refers to the forces of constraint. For by § 91 the forces which keep the z-axis fixed furnish no moment of momentum about this axis. Hence we have for the z-component of (446):

$$\Sigma m_1\left(x_1\frac{d^2y_1}{dt^2} - y_1\frac{d^2x_1}{dt^2}\right) = N_z \quad . \quad . \quad . \quad (468)$$

where the summation is to be performed over all the individual point-masses or elements of mass of the body; and this equation suffices to determine the motion. We have only to express all the variables that occur in it in terms of the single independent variable, which determines the position of the body; for this we shall take the angle ϕ, which an arbitrarily selected plane fixed in the body and passing through the z-axis makes with the xz-plane. Then, if we introduce cylindrical co-ordinates, as in (326a):

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -\rho_1\sin\phi_1\frac{d\phi}{dt} \\ \frac{dy_1}{dt} &= \rho_1\cos\phi_1\frac{d\phi}{dt} \\ \frac{dz_1}{dt} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (469)$$

where ρ_1 denotes the constant distance of the point 1 from the z-axis; and by differentiating once again, we have:

$$\left.\begin{aligned} \frac{d^2x_1}{dt^2} &= -\rho_1\cos\phi_1\left(\frac{d\phi}{dt}\right)^2 - \rho_1\sin\phi_1\frac{d^2\phi}{dt^2} \\ \frac{d^2y_1}{dt^2} &= -\rho_1\sin\phi_1\left(\frac{d\phi}{dt}\right)^2 + \rho_1\cos\phi_1\frac{d^2\phi}{dt^2} \\ \frac{d^2z_1}{dt^2} &= 0 \end{aligned}\right\} \quad . \quad (469a)$$

and consequently, by substituting in (468):

$$\frac{d^2\phi}{dt^2}\Sigma m_1{\rho_1}^2 = N_z \quad . \quad . \quad (469b)$$

This equation of motion has exactly the form of that (8) of a point-mass moving in a straight line, except that the acceleration is replaced by the angular acceleration, the force by the moment of momentum and the constant inertial mass by the constant sum $\Sigma m_1 \rho_1{}^2$, which last is therefore called the *moment of inertia* J of the body with respect to the z-axis.

The equation is therefore integrated by methods perfectly similar to those used in the case of rectilinear motion.

The method of derivation is still more direct if we use Lagrange's equations of the second kind. For the work of the external forces in a displacement of the body about the angle $d\phi$ is by (327):

$$A = \boldsymbol{N}_z \cdot d\phi,$$

and so the external component of force is, by (403):

$$\Phi = \boldsymbol{N}_z.$$

On the other hand if we use (469) the Lagrange function is:

$$L = K = \frac{1}{2}\Sigma m_1 q_1{}^2 = \frac{1}{2} J \dot{\phi}^2 \quad . \quad . \quad . \quad (470)$$

and so, by (405):

$$J\ddot{\phi} = \boldsymbol{N}_z \quad . \quad . \quad . \quad (471)$$

as above.

§ 141. Now that the motion is determined by means of (471) we get for the resultant force and the resultant moments of momentum of the forces of constraint by which the axis of rotation is kept fixed or, what comes to the same thing, for the resistance which the axis of rotation must offer, the following five equations, from (445) and (446):

$$\boldsymbol{F}' = \Sigma m_1 \ddot{\boldsymbol{r}}_1 - \boldsymbol{F} \quad . \quad . \quad . \quad . \quad (472)$$

$$\left.\begin{aligned} \boldsymbol{N}_x' &= \Sigma m_1\left(y_1 \frac{d^2 z_1}{dt^2} - z_1 \frac{d^2 y_1}{dt^2}\right) - \boldsymbol{N}_x \\ \boldsymbol{N}_y' &= \Sigma m_1\left(z_1 \frac{d^2 x_1}{dt^2} - x_1 \frac{d^2 z_1}{dt^2}\right) - \boldsymbol{N}_y \end{aligned}\right\} \quad . \quad (473)$$

We shall here investigate the special case where the driving forces are all zero—that is, where the body rotates with a fixed angular velocity $\dot{\phi}$ about the fixed z-axis.

Then $\boldsymbol{F}$ and $\boldsymbol{N}$ are both zero, whereas the forces of constraint $\boldsymbol{F}'$ and $\boldsymbol{N}'$ are determined by the last equations, if we omit the terms in $\boldsymbol{F}$ and $\boldsymbol{N}$ in them.

The resultant force of constraint $\boldsymbol{F}'$ may be clearly visualized. For if we substitute in (472) for the component of $\boldsymbol{r}_1$ the values (469*a*) we get, in view of $\ddot{\phi} = 0$ and (287):

$$\begin{aligned}\boldsymbol{F}_x' &= -\dot{\phi}^2 \Sigma m_1 \rho_1 \cos\phi_1 = -\dot{\phi}^2 \Sigma m_1 x_1 = -\dot{\phi}^2 x_0 \Sigma m_1,\\ \boldsymbol{F}_y' &= -\dot{\phi}^2 y_0 \Sigma m_1,\\ \boldsymbol{F}_z' &= 0.\end{aligned}$$

That is, the resultant force of constraint is equal and opposite to the centrifugal force of the centre of gravity rotating with the mass Σm_1 and the angular velocity ϕ about the axis of rotation—a theorem which also follows directly from the principle of the motion of the centre of gravity (§ 131).

If we further assume that the axis of rotation passes through the centre of gravity, the resultant force of constraint $\boldsymbol{F}$ vanishes entirely, but the moments of momentum of the forces of constraint $\boldsymbol{N}_x'$ and $\boldsymbol{N}_y'$ in general differ from zero, being:

$$\left.\begin{aligned}\boldsymbol{N}_x' &= \dot{\phi}^2 \Sigma m_1 y_1 z_1\\ \boldsymbol{N}_y' &= -\dot{\phi}^2 \Sigma m_1 x_1 z_1\end{aligned}\right\} \qquad (474)$$

This signifies that even in the present case the axis of rotation must be supported by external forces—namely, by a couple, if it is to remain at rest; or that once the axis is released the body, although no driving forces are acting and although the centre of gravity remains at rest, will no longer be able to preserve its direction of rotation. The question as to how the axis of rotation changes belongs to a later investigation where we deal with motion about a fixed point.

It is only in the special case where:

$$\Sigma m_1 y_1 z_1 = 0 \text{ and } \Sigma m_1 x_1 z_1 = 0 \quad . \quad . \quad (475)$$

that the external constraint vanishes entirely in the rotation in question, and that the z-axis has the property of being a " free " or a " permanent " axis of rotation.

§ 142. Having been led in § 140 to form the concept of the moment of inertia $\Sigma m\rho^2 = J$ of a body with respect to a definite axis, we shall now investigate the question more closely as to the laws connecting the magnitude of the moment of inertia of a definite body with the position of the axis. For the latter may be chosen quite arbitrarily from the outset and mav even lie quite outside the mass of the body.

Let us first consider the moments of inertia of a definite body with respect to all such straight lines as pass through a single point, the origin of co-ordinates O. The position of such a straight line is determined by its direction cosines λ, μ, ν. Now, if x, y, z denote the co-ordinates of the point-mass m, and r its distance from O, then :

$$\rho = r \cdot \sin\theta,$$

where θ denotes the angle between the radius vector r and the straight line (λ, μ, ν); thus :

$$\cos\theta = \lambda \cdot \frac{x}{r} + \mu \cdot \frac{y}{r} + \nu \cdot \frac{z}{r}.$$

Consequently, the moment of inertia of the body with respect to the straight line (λ, μ, ν) has the value :

$$J = \Sigma mr^2 \sin^2\theta = \Sigma mr^2 (\lambda^2 + \mu^2 + \nu^2 - \cos^2\theta)$$

$$J = \lambda^2 \cdot \Sigma m (y^2 + z^2) + \mu^2 \Sigma m (z^2 + x^2) + \nu^2 \Sigma m (x^2 + y^2) - 2\mu\nu \Sigma myz - 2\nu\lambda \Sigma mzx - 2\lambda\mu \Sigma mxy \quad . \quad (476)$$

If we now allow the direction of the straight line, and hence λ, μ, ν to vary, the six sums Σ remain constant, and therefore show that they are characteristic for the magnitudes of the moments of inertia with respect to all the straight lines that pass through O. The last three sums are the moments of inertia J_x, J_y, J_z with respect to the three co-ordinate axes. The last three are also called the " moments of deviation."

A good idea of the manner in which the quantity J

depends on λ, μ, ν may be obtained if we imagine the value of $\frac{1}{\sqrt{J}}$ to be marked off as a distance from O on each direction λ, μ, ν passing through O. The end-points ξ, η, ζ of all these distances then constitute a surface, whose equation is determined by the relationships :

$$\xi = \frac{\lambda}{\sqrt{J}}, \quad \eta = \frac{\mu}{\sqrt{J}}, \quad \zeta = \frac{\nu}{\sqrt{J}}$$

together with (476).

Eliminating λ, μ, v from these four equations we get for the equation to the surface :

$$\xi^2 J_x + \eta^2 J_y + \zeta^2 J_z - 2\eta\zeta\Sigma myz - 2\zeta\xi\Sigma mzx - 2\xi\eta\Sigma mxy = 1 \quad . \quad (477)$$

that is, an ellipsoid whose centre is at the origin; it is called the " ellipsoid of inertia " of the body with respect to the point O.

Every point in the whole of infinite space may be regarded as the centre of such an ellipsoid of inertia, and the moment of inertia of the body with respect to any straight line which passes through this point is equal to the reciprocal of the square of the corresponding semi-diameter of the ellipsoid.

Since the magnitude of the moment of inertia is independent of the choice of co-ordinate axes, the form of the ellipsoid of inertia is also independent of this choice.

The principal axes of the ellipsoid are called the " principal axes of inertia," and the corresponding moments of inertia the " principal moments of inertia." The latter are at the same time the reciprocals of the squares of the semi-axes of the ellipsoid.

The equation to the ellipsoid of inertia assumes a particularly simple form if we allow the co-ordinate axes to coincide with the principal axes of inertia; in that case it runs :

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1,$$

or, if we denote the principal moments of inertia by P, Q, R:

$$P\xi^2 + Q\eta^2 + R\zeta^2 = 1 \quad . \quad . \quad . \quad (478)$$

Comparison with the general equation (477) shows that when the principal axes of inertia are the co-ordinate axes the deviation moments vanish:

$$\Sigma myz = 0, \quad \Sigma mzx = 0, \quad \Sigma mxy = 0 \quad . \quad (479)$$

and from this it again follows by (476) that the moment of inertia of the body with respect to a straight line which makes direction-cosines λ, μ, v with the principal axes of inertia are:

$$J = P\lambda^2 + Q\mu^2 + R\nu^2 \quad . \quad . \quad . \quad (480)$$

The greatest of the three principal moments of inertia P, Q, R, namely that which belongs to the smallest axis of the ellipsoid of inertia, at the same time represents the greatest moment of inertia which a straight line through O can have at all, and conversely. If in particular $P = Q = R$, then $J = P$, and the ellipsoid of inertia becomes a sphere. This holds, for example, for the centre of a homogeneous body in the form of a sphere or a cube, on grounds of symmetry.

The position, the magnitude and the direction of the axes of the ellipsoid become changed if the centre of the ellipsoid is displaced. In general it may be said that the dimensions of the ellipsoid of inertia contract the more the further the centre of the ellipsoid is away from the body.

If we compare the equations (479) and (475), we find that the principal axes of inertia which pass through the centre of gravity of the body have at the same time the property of free or permanent axes of rotation, and are the only ones of their kind to have this property.

§ 143. We next inquire into the moments of inertia of a body with respect to two straight lines, which do not start out from the same point. We shall begin with two parallel lines. If we take one as the z-axis, we may take the second as the z'-axis of a second accented co-ordinate-system with its axes parallel to the first. We then obtain for the two moments of inertia that are to be compared :

$$J_z = \Sigma m(x^2 + y^2),\ J_{z'} = \Sigma m(x'^2 + y'^2)$$

Without loss of generality we may further take the plane through z and z' as the xz-plane and the origin O' to be on the x-axis. The equations of transformation then simply run :

$$x' = x - h,\ y' = y,\ z' = z,$$

where h denotes the distance between the two parallels ; the moment of inertia with respect to the z'-axis becomes :

$$J_{z'} = \Sigma m(x^2 + y^2) - 2h\Sigma mx + h^2\Sigma m.$$

If we now allow the z-axis to pass through the centre of gravity of the body, $\Sigma mx = 0$, and the last equation runs :

$$J_{z'} = J_z + h^2\Sigma m \quad . \quad . \quad . \quad . \quad (481)$$

That is, the moment of inertia of a body with respect to any straight line is equal to the moment of inertia of the body with respect to the parallel which passes through the centre of gravity, increased by the product of the total mass and the square of the distance of the straight line from the centre of gravity (Steiner's theorem). Hence of all the straight lines parallel to a certain direction that which passes through the centre of gravity has the smallest moment of inertia, and the remaining parallels group themselves according to their moment of inertia in coaxial circular cylinders about this one straight line.

This at the same time gives the answer to the general question as to the moment of inertia J of a body with respect to any arbitrary straight line. For if M denotes the mass, P, Q, R the principal moments of inertia with respect to its centre of gravity, then by (460) and (481) :

$$J = P\lambda^2 + Q\mu^2 + R\nu^2 + Mh^2. \quad . \quad . \quad (482)$$

where λ, μ, ν denote the direction-cosines of the straight line with respect to the principal axes of inertia, and h its distance from the centre of gravity.

§ 144. We shall now apply our results to the motion of a heavy body which has a fixed horizontal axis of rotation, a so-called "physical pendulum," in contrast with the "mathematical pendulum" treated in § 69. In addition to the notation there used, we define the position of the body by the angle ϕ which the plane through the axis of rotation and the centre of gravity S of the body makes with the vertical plane through the axis of rotation, and take as the plane of the diagram (Fig. 38) the plane through S and perpendicular to the axis of rotation, which intersects it at the point O. Then the driving force Mg has the point of application S, and its turning moment with respect to the fixed axis is:

$$- Mgh \sin \phi,$$

where $h = SO$ is the distance of the centre of gravity from the axis. Accordingly, by (471):

$$J \frac{d^2\phi}{dt^2} = - Mgh \sin \phi \quad . \quad . \quad . \quad (483)$$

If we compare this equation with the equation (244) for a mathematical pendulum of length l we see that they become fully identical if we set:

$$l = \frac{J}{Mh} \quad . \quad . \quad . \quad . \quad (484)$$

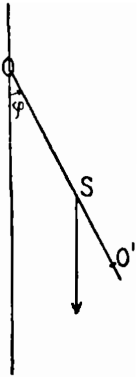

FIG. 38.

That is, the motion of a physical pendulum takes place exactly like a mathematical pendulum of length l determined by (484). Hence this quantity is called the "equivalent or reduced length of the pendulum," and the point at a distance l from O on the straight line OS is called the "vibration centre" or "centre of oscillation" O'.

The way in which the distances l and h depend on each other is ascertained as follows.

By (481) we have:

$$J = J_0 + Mh^2,$$

where J_0 is the moment of inertia of the body with respect to the straight line which passes through the centre of gravity S and is parallel to the axis of rotation. By substituting in (484) we get:

$$l = h + \frac{J_0}{Mh} \quad . \quad . \quad . \quad . \quad . \quad . \quad (485)$$

hence $l > h$, as depicted in Fig. 38. If we displace the axis of rotation to another parallel straight line, then h and l change in accordance with (485), whereas J_0 and M remain constant.

Even when h becomes very small, as well as when it becomes very great (centre of gravity very near to or very far from the axis of rotation) the reduced length of the pendulum l and with it the time of vibration assume very great values; in the latter case the centre of gravity and the centre of vibration come close together as in the case of the mathematical pendulum. There is a minimum of l for $h = \sqrt{\frac{J_0}{M}}$, namely $l = 2\sqrt{\frac{J_0}{M}} = 2h$.

If we start from any arbitrary value of h, to which a certain value of l corresponds, and displace the axis of rotation to the parallel through the centre of gravity O', so that we make:

$$h' = l - h = \frac{J_0}{Mh},$$

then we get as the new centre of vibration the point which is at a distance from O' given by:

$$l' = h' + \frac{J_0}{Mh'} = l - h + h = l \quad . \quad . \quad (486)$$

That is, the new centre of vibration coincides with O and the reduced length of the pendulum is the same as before. The importance of the *reversible pendulum* depends on this theorem.

§ 145. We next consider the motion of a rigid body about a *fixed point*, which brings us to an essentially new class of phenomena. For whereas the rotation of a body about a fixed straight line, as we have seen, exhibits a certain analogy with the motion of a point on a circular arc (physical and mathematical pendulum), the rotation about a fixed point is essentially more complicated than the motion of a point on a spherical surface, because here there are three degrees of freedom as contrasted with two in the former case.

Let us therefore first get a purely kinematic picture of the nature of such a rotation.

We know from § 101 that the rotation about a fixed point O is at every moment a rotation about a straight line which passes through O. This straight line will not, however, be the same for all times, but will change its direction continuously, both in space and in the body—that is, not only the angles will change, which the instantaneous axis of rotation makes with the co-ordinate axes, but also the material points which represent the axis of rotation.

To obtain a clear picture of this we may first imagine that the rotation occurs for a finite but extremely small time about the instantaneous axis of rotation at each moment, and then suddenly changes abruptly to a very closely neighbouring axis of rotation.

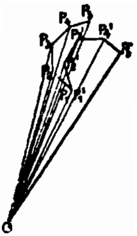

FIG. 39.

The directions which assume the part of the axis of rotation then form a succession of straight lines in space: OP_1, OP_2, OP_3, . . . (Fig. 39).

On the other hand, those material straight lines, about which the rotation occurs in succession, form a sequence of straight lines in the body: OP_1', OP_2', OP_3', . . . which we may fix by assigning symbols to their points of intersection P_1', P_2', P_3', . . . with the surface of the body. When these two sequences of straight lines, those fixed in space and those fixed in the body, are known, we arrive directly at the nature of the motion: it is a rotation of the body, and hence also of the sequence of straight lines OP' that lie in the body, about the straight line which is at each moment in question common to the two sequences; at the moment taken in the diagram (Fig. 39) this is a rotation about the straight line OP_1. The abrupt transition to the next axis of rotation always occurs at the exact moment when the next straight line of the sequence OP' coincides with the next straight line of the sequence OP—in the figure, when OP_2' coincides with OP_2. In this way the axes of rotation OP_1, OP_2, OP_3, . . . come into action successively, so soon as the material straight lines OP_1', OP_2', OP_3', . . . come to lie in their directions.

If we finally pass from the finite small times and angles to the infinitely small values, the two sequences of straight lines become transformed into two cones OP and OP' which touch along a straight line, the first of which is fixed in space and the second is fixed in the body. The motion of the body about the fixed point O is shown to be identical with the rolling of the cone which is fixed in the body on the cone which is fixed in space, the term "rolling" being applied to the motion where the straight line of contact of the two cones remains at rest.

If one of the cones contracts to a single straight line, so does the other. The rotation then occurs about an axis which is fixed in space as well as in the body.

§ 146. The dynamical laws of the motion are completely contained in the three equations (463):

$$\Sigma[\mathbf{r}, m\ddot{\mathbf{r}}] = \mathbf{N} \quad . \quad . \quad . \quad . \quad . \quad (487)$$

where $\mathbf{N}$ denotes the moment of momentum of the driving forces with respect to the fixed point O.

For the forces of constraint which keep this point fixed are unable to contribute a moment of momentum about it. The difficulty of the problem consists only in referring the sum Σ to the three independent variables on which the position of the body depends, and to their first and second differential coefficients with respect to the time. We shall here attack this problem directly.

To deal with the position of the body we follow the method given in § 100, in order not to lose the advantage of symmetry, and introduce an accented co-ordinate system, which is fixed in the body, by means of the equations (329). There are then six relations between the nine direction-cosines, α_1 to γ_3. These relations may, according to requirements, be expressed in the forms (331), (332), (333), (334), but may also be formulated in other useful ways. The most important of these formulations results if we solve the equations (329) for x', y', z' and then identify them with those equations which result from (329) if we simply exchange the unaccented quantities in it with the accented quantities and at the same time the letters α, β, γ with the digits 1, 2, 3.

It then follows that :

$$\alpha_1 = \frac{\beta_2\gamma_3 - \beta_3\gamma_2}{D}, \quad \beta_1 = \frac{\gamma_2\alpha_3 - \gamma_3\alpha_2}{D},$$
$$\gamma_1 = \frac{\alpha_2\beta_3 - \alpha_3\beta_2}{D}, \text{ and so forth} \quad . \quad (488)$$

D denotes the determinant of the equations, namely :

$$D = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (489)$$

This determinant has a very simple value. For if we square and add the three equations (488), we get by (331) :

$$\begin{aligned} D^2 &= (\beta_2\gamma_3 - \beta_3\gamma_2)^2 + (\gamma_2\alpha_3 - \gamma_3\alpha_2)^2 + (\alpha_2\beta_3 - \alpha_3\beta_2)^2 \\ &= (\alpha_2^2 + \beta_2^2 + \gamma_2^2)(\alpha_3^2 + \beta_3^2 + \gamma_3^2) - (\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3)^2 \\ &= 1 \end{aligned}$$

Hence :

$$D = \pm 1 \quad . \quad . \quad . \quad . \quad . \quad (490)$$

The sign of D is determined by considering a special case. For since the direction-cosines change continuously, D is also continuous and in view of (490) is absolutely constant. If we now make the x'-axis coincident with the x-axis and the y'-axis coincident with the y-axis ($\alpha_1 = 1$, $\beta_2 = 1$), then the z'-axis also coincides with the z-axis ($\gamma_3 = +1$), since quite early, in § 100, we also assumed the accented co-ordinate system to be right-handed. Accordingly the determinant (489) is:

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad . \qquad . \quad (491)$$

and retains this value for all positions of the fixed system in the body.

Hence the equations (488) become:

$$\alpha_1 = \beta_2\gamma_3 - \beta_3\gamma_2 \quad . \quad . \quad . \quad . \quad (492)$$

and analogous expressions for each of the other eight direction-cosines.

The law according to which these nine relations are formed becomes clear immediately if we observe that on the right-hand side of the equation only those letters and digits occur which are absent on the left-hand side.

For example:

$$\gamma_2 = \alpha_3\beta_1 - \alpha_1\beta_3, \text{ and so forth.}$$

§ 147. Let us next consider the state of the body as regards *velocities*. Here, too, we link up with the result derived in Chapter II, that the most general infinitesimal displacement of the body is represented by the components of a rotation with respect to the three co-ordinate axes. We also adopt the notation of § 100, but shall now take ξ, η, ζ to stand, not for the infinitesimal angles of rotation themselves, but for the finite ratios of these angles to the element of time dt, which we find it appropriate to call the *components of the velocity of rotation* with respect to the co-ordinate axes x, y, z. Instead of the equations (336) we then get:

$$\left.\begin{aligned} \xi &= \beta_1\frac{d\gamma_1}{dt} + \beta_2\frac{d\gamma_2}{dt} + \beta_3\frac{d\gamma_3}{dt} = -\left(\gamma_1\frac{d\beta_1}{dt} + \gamma_2\frac{d\beta_2}{dt} + \gamma_3\frac{d\beta_3}{dt}\right) \\ \eta &= \gamma_1\frac{d\alpha_1}{dt} + \gamma_2\frac{d\alpha_2}{dt} + \gamma_3\frac{d\alpha_3}{dt} = -\left(\alpha_1\frac{d\gamma_1}{dt} + \alpha_2\frac{d\gamma_2}{dt} + \alpha_3\frac{d\gamma_3}{dt}\right) \\ \zeta &= \alpha_1\frac{d\beta_1}{dt} + \alpha_2\frac{d\beta_2}{dt} + \alpha_3\frac{d\beta_3}{dt} = -\left(\beta_1\frac{d\alpha_1}{dt} + \beta_2\frac{d\alpha_2}{dt} + \beta_3\frac{d\alpha_3}{dt}\right) \end{aligned}\right\} \quad (493)$$

and instead of the equation (335) we get the analogous equations:

$$\alpha_1 \frac{d\alpha_1}{dt} + \alpha_2 \frac{d\alpha_2}{dt} + \alpha_3 \frac{d\alpha_3}{dt} = 0, \text{ and so forth} \quad . \quad (494)$$

The ratios of the components ξ, η, ζ again determine the direction of the axis of rotation, and the absolute value of the vector determines the value of the velocity of rotation—that is, the ratio of the infinitesimal angle of rotation to the element of time dt:

$$\omega = + \sqrt{\xi^2 + \eta^2 + \zeta^2} \quad . \quad . \quad . \quad (495)$$

As in the case of every vector so also in that of the velocity of rotation the component in any arbitrary direction is obtained by multiplying the three quantities ξ, η, ζ individually by the corresponding direction-cosines of the direction in question, and by adding together the products so obtained. We also obtain according to the same rule the components of the velocity of rotation in those directions which the accented co-ordinates assume at the moment in question:

$$\left.\begin{aligned} \alpha_1\xi + \beta_1\eta + \gamma_1\zeta &= \xi' \\ \alpha_2\xi + \beta_2\eta + \gamma_2\zeta &= \eta' \\ \alpha_3\xi + \beta_3\eta + \gamma_3\zeta &= \zeta' \end{aligned}\right\} \quad . \quad . \quad (496)$$

or, as immediately follows from them:

$$\left.\begin{aligned} \xi &= \alpha_1\xi' + \alpha_2\eta' + \alpha_3\zeta' \\ \eta &= \beta_1\xi' + \beta_2\eta' + \beta_3\zeta' \\ \zeta &= \gamma_1\xi' + \gamma_2\eta' + \gamma_3\zeta' \end{aligned}\right\} \quad . \quad . \quad . \quad (497)$$

The notation has again been chosen in such a way that the unaccented quantities ξ, η, ζ correspond to the letters α, β, γ and the accented quantities to the digits 1, 2, 3.

We usually call ξ', η', ζ' the components of the velocity of rotation of the body with respect to the accented co-ordinate axes. But this term is to be used with caution, since the body is fixed in the accented co-ordinate system and hence has the velocity of rotation zero with respect to it throughout.

Introducing the components of the velocity of rotation gives us the important advantage that the time differential coefficients of all nine direction-cosines $\alpha_1, \ldots \gamma_3$ may be expressed conveniently and symmetrically in terms of these three quantities. Hence, for example, by (496) and (492):

$$\alpha_1\eta' - \alpha_2\xi' = \alpha_1(\alpha_2\xi + \beta_2\eta + \gamma_2\zeta) - \alpha_2(\alpha_1\xi + \beta_1\eta + \gamma_1\zeta)$$
$$= \gamma_3\eta - \beta_3\zeta \quad . \quad . \quad . \quad . \quad . \quad (497a)$$

and further, by 493:

$$= \gamma_3\left(\gamma_1\frac{d\alpha_1}{dt} + \gamma_2\frac{d\alpha_2}{dt} + \gamma_3\frac{d\alpha_3}{dt}\right) + \beta_3\left(\beta_1\frac{d\alpha_1}{dt} + \beta_2\frac{d\alpha_2}{dt} + \beta_3\frac{d\alpha_3}{dt}\right)$$

$$= -\alpha_1\alpha_3\frac{d\alpha_1}{dt} - \alpha_3\alpha_2\frac{d\alpha_2}{dt} + (1 - \alpha_3^2)\frac{d\alpha_3}{dt} = \frac{d\alpha_3}{dt}.$$

Therefore:

$$\frac{d\alpha_3}{dt} = \alpha_1\eta' - \alpha_2\xi' \quad . \quad . \quad . \quad . \quad (498)$$

and, correspondingly, eight other relations; the law according to which these expressions are constructed is characterized by the fact that on the right-hand side of the equation only that letter occurs which also appears on the left-hand side (α in (498)), whereas conversely just those digits occur on the right which are missing on the left (1 and 2 in (498)).

Parallel with these nine relations we have, by (497a), the other nine of the form:

$$\frac{d\alpha_3}{dt} = \gamma_3\eta - \beta_3\zeta, \text{ and so forth} \quad . \quad . \quad (499)$$

in which the letters α, β, γ and the digits 1, 2, 3 have exchanged rôles.

The perfectly analogous construction of the accented and the unaccented components of the velocity of rotation is also shown in the relationships by which the ξ', η', ζ''s are expressed in terms of the differential coefficients of the α, β, γ's. We arrive at them by combining (496), (492) and (499):

$$\begin{aligned}\xi' &= (\beta_2\gamma_3 - \beta_3\gamma_2)\,\xi + (\gamma_2\alpha_3 - \gamma_3\alpha_2)\,\eta + (\alpha_2\beta_3 - \alpha_3\beta_2)\,\zeta \\ &= \alpha_2(\beta_3\zeta - \gamma_3\eta) + \beta_2(\gamma_3\xi - \alpha_3\zeta) + \gamma_2(\alpha_3\eta - \beta_3\xi) \\ \xi' &= -\left(\alpha_2\frac{d\alpha_3}{dt} + \beta_2\frac{d\beta_3}{dt} + \gamma_2\frac{d\gamma_3}{dt}\right) \\ &= \alpha_3\frac{d\alpha_2}{dt} + \beta_3\frac{d\beta_2}{dt} + \gamma_3\frac{d\gamma_2}{dt}, \text{ and so forth} \quad . \quad . \qquad (500)\end{aligned}$$

which is completely analogous to (493).

Corresponding to the three equations (494) there are the following three:

$$\alpha_1\frac{d\alpha_1}{dt} + \beta_1\frac{d\beta_1}{dt} + \gamma_1\frac{d\gamma_1}{dt} = 0, \text{ and so forth} \quad . \qquad (501)$$

§ 148. We are now sufficiently well prepared to reduce the three equations (487) directly to terms of the independent variables. We write the first of them in the form:

$$\frac{d}{dt}\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = N_x \quad . \quad . \qquad (502)$$

and introduce in place of the unaccented co-ordinates x, y, z of the point mass m the accented co-ordinates, since the latter do not depend on the time. We accomplish this by means of the equations (329) and their derivatives:

$$\frac{dx}{dt} = x'\frac{d\alpha_1}{dt} + y'\frac{d\alpha_2}{dt} + z'\frac{d\alpha_3}{dt} \quad . \quad . \quad . \quad . \qquad (503)$$

and we get for the sum in (502) a number of terms in which the direction-cosines and their derivatives can be placed in front of the summation sign Σ. The six quantities $\Sigma m x'^2$, $\Sigma m y'^2$, $\Sigma m z'^2$, $\Sigma m x'y'$, $\Sigma m y'z'$, $\Sigma m z'x'$ alone remain after the sign, and they are all independent of the time.

If from now on we allow the accented co-ordinate axes to coincide with the principal axes of inertia of the body with respect to the fixed point O, then by (479) the last three sums vanish and of (502) there only remains :

$$\frac{d}{dt}\left\{\left(\beta_1\frac{d\gamma_1}{dt}-\gamma_1\frac{d\beta_1}{dt}\right)\Sigma mx'^2+\left(\beta_2\frac{d\gamma_2}{dt}-\gamma_2\frac{d\beta_2}{dt}\right)\Sigma my'^2\right.$$
$$\left.+\left(\beta_3\frac{d\gamma_3}{dt}-\gamma_3\frac{d\beta_3}{dt}\right)\Sigma mz'^2\right\}=N_x$$

or if we introduce ξ', η', ζ' from the relations (498) and take into account (492) :

$$\frac{d}{dt}\{(\alpha_2\eta'+\alpha_3\zeta')\,\Sigma mx'^2+(\alpha_3\zeta'+\alpha_1\xi')\,\Sigma my'^2+$$
$$(\alpha_1\xi'+\alpha_2\eta')\,\Sigma mz'^2\}=N_x$$

and finally, if we again, as in § 142, denote the principal moments of inertia by P, Q, R and correspondingly write the three components of the velocity of rotation referred to the principal axes of inertia thus :

$$\xi'=p,\quad \eta'=q,\quad \zeta'=r, \quad . \quad . \quad . \qquad (504)$$

then we have :

$$\frac{d}{dt}(\alpha_1pP+\alpha_2qQ+\alpha_3rR)=N_x$$

likewise :

$$\frac{d}{dt}(\beta_1pP+\beta_2qQ+\beta_3rR)=N_y \quad . \quad . \qquad (505)$$

and :

$$\frac{d}{dt}(\gamma_1pP+\gamma_2qQ+\gamma_3rR)=N_z$$

Although these three equations are constructed very simply and compactly, they have the disadvantage that the nine direction-cosines occur in them. We can get rid of them by multiplying the equations in turn by the corresponding direction-cosines and adding them together —that is, by referring the equations of motion to the accented co-ordinate axes instead of to the unaccented co-ordinate axes. We then get, by first multiplying by

α_1, β_1, γ_1 and afterwards adding, taking into account (501), (500) and (504):

$$P\frac{dp}{dt} - (Q - R)qr = N_{x'}$$

similarly:

$$Q\frac{dq}{dt} - (R - P)rp = N_{y'} \qquad (506)$$

and:

$$R\frac{dr}{dt} - (P - Q)pq = N_{z'}$$

These are called Euler's equations of motion. Their characteristic feature is the second term on the left-hand side, by which they are distinguished from the equation for the rotation about a fixed axis, and which cause the perturbations of the axis of rotation.

§ 149. We shall now consider some special applications, the first being the case where the body is initially at rest—that is, the initial values p_0, q_0, r_0 all vanish. The question is: about which straight line will the body begin rotating under the action of the given forces?

Since the direction ratios of the axis of rotation with respect to the principal axes of inertia are represented in general by the ratios $p : q : r$, and since for a sufficiently short time t:

$$p = p_0 + \left(\frac{dp}{dt}\right)_0 \cdot t = \left(\frac{dp}{dt}\right)_0 \cdot t,$$

we have at the beginning of the motion:

$$p : q : r = \left(\frac{dp}{dt}\right)_0 : \left(\frac{dq}{dt}\right)_0 : \left(\frac{dr}{dt}\right)_0$$

and by (506):

$$p : q : r = \frac{N_{x'}}{P} : \frac{N_{y'}}{Q} : \frac{N_{z'}}{R} \qquad (507)$$

This equation also completely disposes of the question which was left unanswered at the end of § 131 concerning the nature of the motion into which a couple sets a free rigid body originally at rest. We there saw that the centre of gravity of the body remains at rest; here we have found the direction of the initial axis of rotation. This direction coincides with the direction of the couple N only if either the three principal moments of inertia are equal to one another or if the axis of the couple is a principal axis of inertia. Then the two other components of N are equal to zero.

In general, the relationship between the direction of the couple and the direction of the initial axis of rotation can be clearly visualized by means of the ellipsoid of inertia. For the direction of the axis of rotation is the conjugate diameter to the plane of the couple with respect to the ellipsoid—that is, it is that diameter of the ellipsoid at whose end-point the tangential plane is parallel to the plane of the couple. For the tangential plane of the ellipsoid (478) at the extremity of the diameter $p : q : r$ has the direction $Pp : Qq : Rr$ as its normal, and by (507) this is also the direction of the axis of the couple N.

§ 150. We next consider the special case where for a given initial state the turning moment N *of the external forces completely vanishes*, as, for example, when the body is supported at its centre of gravity or when no gravitational force acts at all. Then Euler's equations simplify to:

$$P\frac{dp}{dt} - (Q - R)\,qr = 0 \quad . \quad . \quad . \quad (507a)$$

We first inquire into the condition that the rotation shall always occur about the same axis. Then, as we have already seen in § 145, this axis remains fixed both in space and in the body, and the rotation takes place with constant velocity, in accordance with the principle of vis viva. So p, q, r are constant, and, from (507a), it follows that:

$$(Q - R)qr = 0$$

together with the two corresponding equations. These three conditions require either that $P = Q = R$—that is, the central ellipsoid becomes a sphere—or, for any arbitrary body, that two of the components p, q, r be equal to zero—that is, that the motion occurs about a principal axis of inertia—a result which agrees exactly with that derived at the end of § 142 in a much simpler way.

In the general case, with any initial state, the three equations (507a) admit of two simple integrations. By multiplying by p, q, r and subsequently adding up we get, by integrating :

$$Pp^2 + Qq^2 + Rr^2 = c^2 \quad . \quad . \quad . \quad (508)$$

and by multiplying by Pp, Qq, Rr we get in the same way :

$$P^2p^2 + Q^2q^2 + R^2r^2 = c'^2 \quad . \quad . \quad . \quad (509)$$

We easily convince ourselves that (508) expresses the principle of vis viva and (509) the principle of sectorial areas. For according to the former principle the vis viva of the rotation, being the only kind of energy present, is constant; that is, by (470) and (480) :

$$K = \frac{1}{2}J\omega^2 = \frac{1}{2}(P\lambda^2 + Q\mu^2 + R\nu^2)\omega^2$$

$$= \frac{1}{2}(Pp^2 + Qq^2 + Rr^2) = \frac{c^2}{2} \quad . \quad (510)$$

and according to the principle of sectorial areas the equations (505) give, on being integrated :

$$\left.\begin{aligned} \alpha_1 pP + \alpha_2 qQ + \alpha_3 rR &= c'_x \\ \beta_1 pP + \beta_2 qQ + \beta_3 rR &= c'_y \\ \gamma_1 pP + \gamma_2 qQ + \gamma_3 rR &= c'_z \end{aligned}\right\} \quad . \quad . \quad . \quad (511)$$

By § 137 the constants c'_x, c'_y, c'_z are the components of the resultant turning moment with respect to the point O, and their ratios c'_x, c'_y, c'_z give us the fixed direction in space of the axis of the resultant turning moment, which is perpendicular to the invariable plane; and the sum of their squares is, by (511) and (509) :

$$P^2p^2 + Q^2q^2 + R^2r^2 = c'^2_x + c'^2_y + c'^2_z = c'^2 \, . \quad (512)$$

To determine the velocities of rotation p, q, r as functions of the time t, we require in addition to (508) and (509) a third integral; this integral may be obtained without sacrificing symmetry by calculating the values of p^2, q^2, r^2 from the two equations mentioned taken in conjunction with $p^2 + q^2 + r^2 = \omega^2$, thus:

$$p^2 = \frac{QR\omega^2 - c^2(Q + R) + c'^2}{(P - Q)(P - R)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (513)$$

If we now multiply the equations of motion (507*a*) in turn by $\frac{p}{P}$, $\frac{q}{Q}$, $\frac{r}{R}$ and add, we get:

$$p\frac{dp}{dt} + q\frac{dq}{dt} + r\frac{dr}{dt} = \frac{1}{2}\frac{d\omega^2}{dt} = \left(\frac{Q-R}{P} + \frac{R-P}{Q} + \frac{P-Q}{R}\right)pqr \qquad (514)$$

and by substituting the values of p, q, r from (513):

$$\frac{d\omega^2}{dt} = 2\sqrt{(A - \omega^2)\,(B - \omega^2)\,(C - \omega^2)} \quad . \qquad (515)$$

where we have used the abbreviations:

$$A = \left(\frac{1}{Q} + \frac{1}{R}\right)c^2 - \frac{1}{QR}c'^2$$

$$B = \left(\frac{1}{R} + \frac{1}{P}\right)c^2 - \frac{1}{RP}c'^2$$

$$C = \left(\frac{1}{P} + \frac{1}{Q}\right)c^2 - \frac{1}{PQ}c'^2.$$

Accordingly ω^2 is an elliptic function of t; that is, it is periodic, and so, by (513), are the components of the velocity of rotation with respect to the principal axes of inertia.

§ 151. There is a geometrical representation, which can be easily visualized, of this generally rather complicated motion. It is obtained, according to Poinsot, if instead of fixing our attention on the body itself we consider its ellipsoid of inertia, which is rigidly connected with the body and hence rotates with it.

Let us suppose that at any moment chosen at random the axis of rotation is OP', where P' denotes its point of intersection with the ellipsoid, the so-called "pole of the motion," which has the co-ordinates x', y', z' (Fig. 40). Then by (478):

$$Px'^2 + Qy'^2 + Rz'^2 = 1 \quad . \quad . \quad . \quad (516)$$

and the direction-cosines of the axis of rotation, referred to the axes of the ellipsoid, are:

$$x' : y' : z' = p : q : r, \text{ where } p^2 + q^2 + r^2 = \omega^2.$$

Hence if we set the length of the semi-diameter $OP' = \rho$, we get:

$$x' = \frac{p}{\omega} \cdot \rho,\ y' = \frac{q}{\omega} \cdot \rho,\ z' = \frac{r}{\omega}\rho \quad . \quad (517)$$

and from (516):

$$Pp^2 + Qq^2 + Rr^2 = \frac{\omega^2}{\rho^2}.$$

In conjunction with (508) this gives:

$$\omega = c\rho \quad . \quad . \quad (518)$$

That is, the velocity of rotation is always proportional to the length of the semi-diameter which happens to represent the axis of rotation at the moment in question.

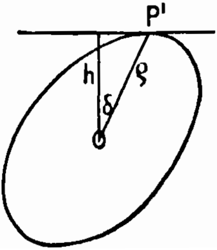

FIG. 40.

Further, according to (478) the normal of the ellipsoid at the pole P' of the rotation has the direction ratios:

$$Px' : Qy' : Rz' = Pp : Qq : Rr$$

The direction-cosines are themselves therefore, in virtue of (509):

$$\frac{Pp}{c'},\ \frac{Qq}{c'},\ \frac{Rr}{c'} \quad . \quad . \quad (519)$$

Thus the normal of the ellipsoid at P' in general changes its direction with respect to the principal axes of the ellipsoid. But if we multiply these three direction-cosines by α_1, α_2, α_3, we get, by (511), a constant—that is, the angle which this normal makes with the unaccented x-axis which is fixed in space is constant, and likewise the angle with the y-axis and the z-axis. Hence the direction of the normal remains fixed in space; by § 137 its direction is nothing else than the normal to the invariable plane; this plane is accordingly parallel to the tangential plane to the ellipse at P'.

But still more. The tangential plane to the ellipsoid of inertia at the pole P' of the rotation not only always remains parallel to itself, but also remains fixed in space. For if we calculate its distance h from the point of rotation O (Fig. 40), we get:

$$h = \rho \cdot \cos \delta$$

where δ denotes the angle between the diameter OP' and the normal to the ellipsoid at P'. Hence by (517) and (519):

$$\cos \delta = \frac{p}{\omega} \cdot \frac{Pp}{c'} + \frac{q}{\omega} \cdot \frac{Qq}{c'} + \frac{r}{\omega} \cdot \frac{Rr}{c'}$$

and in view of (508) and (518):

$$h = \rho \cdot \frac{c^2}{\omega c'} = \frac{c}{c'}, \qquad \ldots \qquad (520)$$

which is constant.

If we recapitulate these theorems we get the following simple picture of the motion. The ellipsoid of inertia rotates about its fixed centre in such a way that it rolls along on a definite fixed tangential plane (§ 145), the velocity of rotation always being proportional to the distance of the point of contact—that is, the pole of the rotation, from the centre.

We again observe here that the axis of rotation preserves a constant direction only if it coincides with a principal axis of inertia, for in every other case the nature of the curvature of the ellipsoid forces the axis of rotation OP' into ever new positions.

We shall enter a little further into the question of the axes of rotation for the present case by linking up with the discussion of § 145. The conditions become clearly visualized if we imagine the fixed (invariable) plane to be blackened, say, with lamp black, so that it leaves its trace where it has made contact with the ellipsoid. Then the black points P' on the ellipsoid define the fixed cone in the body, and the points P on the invariable plane which have parted with their lamp-black define the cone of the axes of rotation which is fixed in space.

Let us first consider the points P' on the ellipsoid; they form the so-called "polhode." Its co-ordinates x', y', z' satisfy not only the equation (516) but in virtue of (517), (509), (518), also the equation:

$$P^2x'^2 + Q^2y'^2 + R^2z'^2 = \frac{1}{h^2} \quad . \quad . \quad . \quad (521)$$

Both equations combined give:

$$P(Ph^2 - 1)x'^2 + Q(Qh^2 - 1)y'^2 + R(Rh^2 - 1)z'^2 = 0 \quad . \quad (522)$$

This is the equation to the cone of the axes of rotation, which is fixed in the body and which cuts the polhode out of the ellipsoid of inertia. It is a cone of the second order, and so the polhode is a closed curve in a compact form. The shape of the cone depends for a definite body on a single parameter, the quantity h^2, which by (520), (508) and (509), has the value:

$$h^2 = \frac{Pp^2 + Qq^2 + Rr^2}{P^2p^2 + Q^2q^2 + R^2r^2} \quad . \quad . \quad . \quad (523)$$

To obtain a clear picture we shall choose the notation of the principal moments of inertia as follows:

$$P \geqq Q \geqq R \quad . \quad . \quad . \quad . \quad . \quad (524)$$

so that P corresponds to the shortest and R to the longest axis of the ellipsoid of inertia.

Then:

$$Ph^2 \geqq 1, \; Qh^2 \gtreqless 1, \; Rh^2 \leqq 1 \quad . \quad . \quad . \quad (525)$$

as we easily see from (523).

Three cases are to be distinguished, according as h^2 is greater than, less than or equal to $\frac{1}{Q}$.

The corresponding forms of the polhode are indicated in Fig. 41, where the axis of the middle moment of inertia Q is to be imagined perpendicular to the plane of the diagram. The polhode consists of two separate parts which lie symmetrically on both sides of the centre O. According as Qh^2 is greater than or less than 1, the polhode surrounds the R-axis or the P-axis; that is, the axis of the smallest or the axis of the greatest moment of inertia; but it never encloses the axis of the intermediate principal moment of inertia. In the limiting case $Qh^2 = 1$ the cone (522) degenerates into the two planes:

$$\frac{z'}{x'} = \pm\sqrt{\frac{P(P-Q)}{R(Q-R)}} \quad . \quad . \quad . \quad (526)$$

and the polhode consists of two ellipses which intersect each other at the extremities of the intermediate principal axis of inertia (indicated by straight lines in Fig. 41); these are the points of contact of the ellipsoid with those tangential planes which are at the distance $h = \frac{1}{\sqrt{Q}}$ from the centre.

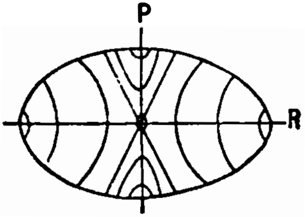

FIG. 41.

Corresponding to the geometrical conditions just described we have the particular features of the physical processes. During the motion of the ellipsoid the pole of the rotation P' advances on the polhode defined by the initial state, but not, of course, in the sense of an ordinary motion of a material point on a curve. What moves is not the matter of the point P'—for this is at rest at the moment of rotation—but rather its property of being the pole of the motion. We here again have a good example of the concept of motion which was described in § 1.

If the pole P' is situated initially at the extremity of a principal axis of inertia, it remains there for all time, corresponding to its property, which has been repeatedly established, of being a free axis of rotation. But we here again recognize an essential difference between the behaviour of the axis of the greatest and that of the least principal moment of inertia as compared with that of the intermediate principal moment of inertia. For if the pole P' does not exactly coincide with the extremity of the P-axis or the R-axis, but only approximately, it remains constantly near this extremity, since the polhode surrounds the axis, as is evident in Fig. 41. But if it deviates from the extremity of the Q-axis ever so little, the polhode on which it pursues its path carries it to great distances from its initial position. For this reason the axes of the greatest and least principal moments of inertia are called "stable" axes of rotation, whereas that of the intermediate principal axis of rotation is called an "unstable" axis of rotation.

The curve of the pole of rotation P on the invariable plane, which defines the cone of the axes of rotation which is fixed in space, is called a "herpolhode"; it is of a more complicated form and, in general, is not closed.

§ 152. Let us now again resume our discussion of the general case of arbitrarily given external forces and besides inquiring into the velocities of rotation p, q, r also inquire into the *position* of the body at the time t. It will often be found desirable to express the position, not in terms of the nine direction-cosines, but by three mutually independent angles, which, of course, entails a sacrifice of symmetry.

We adopt this method of expression here. First we define the direction of the (positive) z'-axis by means of the two polar angles θ (between 0 and π) and ϕ (between 0 and 2π), as in § 32.

A point on the z'-axis at a distance r from the origin then has the co-ordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi = r\cdot\alpha_3 \\ y &= r\sin\theta\sin\phi = r\cdot\beta_3 \\ z &= r\cos\theta \qquad\quad\; = r\cdot\gamma_3. \end{aligned}$$

Thus:

$$\left.\begin{aligned}\alpha_3 &= \sin\theta\cos\phi\\ \beta_3 &= \sin\theta\sin\phi\\ \gamma_3 &= \cos\theta\end{aligned}\right\} \tag{527}$$

The z'-axis having been fixed in this way, the accented co-ordinate system can still turn about this axis. Hence we define the direction of the (positive) x'-axis by means of the angle ψ (between 0 and 2π), which it forms with a fixed direction in the (x', y')-plane, namely with the projection of the (positive) z-axis on this plane, reckoned in the sense of a positive rotation about the z'-axis (Fig. 42).

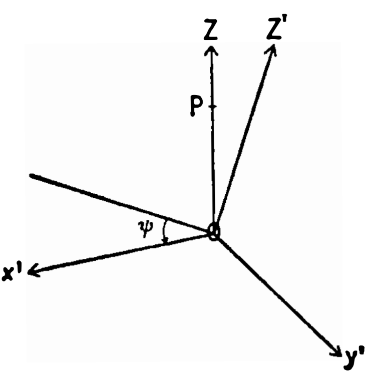

FIG. 42.

A point P on the z-axis at a distance r from the origin then has the co-ordinates:

$$\begin{aligned}x' &= r\sin\theta\cos\psi &&= r\cdot\gamma_1\\ y' &= -r\sin\theta\sin\psi &&= r\cdot\gamma_2\\ z' &= r\cos\theta &&= r\cdot\gamma_3\end{aligned}$$

and so:

$$\left.\begin{aligned}\gamma_1 &= \sin\theta\cos\psi\\ \gamma_2 &= -\sin\theta\sin\psi\end{aligned}\right\}. \tag{528}$$

These expressions then also determine the remaining three direction-cosines.

For from the relationships (492):

$$\alpha_1 = \beta_2\gamma_3 - \beta_3\gamma_2,\ \beta_2 = \gamma_3\alpha_1 - \gamma_1\alpha_3$$

we get:

$$\alpha_1(1-\gamma_3^2) = -\gamma_1\gamma_3\alpha_3 - \gamma_2\beta_3$$

or:

$$\left.\begin{aligned}\alpha_1 &= \sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta\\ \text{Likewise:}\quad \beta_2 &= -\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta\\ \alpha_2 &= \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta\\ \beta_1 &= -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta\end{aligned}\right\}. \tag{529}$$

Now we may also express the components of the velocities of rotation ξ, η, ζ or ξ', η', ζ', respectively, directly in terms of the independent angles ϕ, θ, ψ and their differential coefficients with respect to t, say by means of the relations (493) or (500). We get:

$$\left.\begin{aligned} \xi &= \sin\theta\cos\phi\frac{d\psi}{dt} - \sin\phi\frac{d\theta}{dt} \\ \eta &= \sin\theta\sin\phi\frac{d\psi}{dt} + \cos\phi\frac{d\theta}{dt} \\ \zeta &= \cos\theta\frac{d\psi}{dt} + \frac{d\phi}{dt} \end{aligned}\right\} \quad . \quad (530)$$

$$\left.\begin{aligned} \xi' &= \sin\theta\cos\psi\frac{d\phi}{dt} - \sin\psi\frac{d\theta}{dt} \\ \eta' &= -\sin\theta\sin\psi\frac{d\phi}{dt} - \cos\psi\frac{d\theta}{dt} \\ \zeta' &= \cos\theta\frac{d\phi}{dt} + \frac{d\psi}{dt} \end{aligned}\right\} \quad . \quad . \quad (531)$$

Any of the general relations derived in § 147 may be used to test and confirm these expressions.

The three independent angles ϕ, θ, ψ may also be introduced by applying directly Lagrange's equations of motion of the second kind (405) or Hamilton's canonical equations of motion (412); but on account of lack of symmetry the resulting values are not in a compact form.

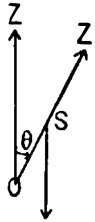

FIG. 43.

§ 153. Finally we shall discuss an example in which a given external force is acting. For this we choose a simple symmetrical *top*, which is supported at a point *O* on its axis of symmetry. We take the axis of symmetry, on which the centre of gravity *S* is also situated, as the z'-axis, but the upward vertical, as usual, as the z-axis (Fig. 43).

If h denotes the distance of the centre of gravity S from the fixed point O, and M the total mass of the top, then the external force is:

$$\boldsymbol{F}_x = 0,\ \boldsymbol{F}_y = 0,\ \boldsymbol{F}_z = -Mg.$$

Its point of application is :

$$x_0 = h\alpha_3,\ y_0 = h\beta_3,\ z_0 = h\gamma_3 \quad . \quad . \quad . \quad (532)$$

and its moment of momentum :

$$\boldsymbol{N}_x = -\ Mgh\beta_3,\ \boldsymbol{N}_y = Myh\alpha_3,\ \boldsymbol{N}_z = 0$$

or :

$$\boldsymbol{N}_{x'} = Mgh\gamma_2,\ \boldsymbol{N}_{y'} = -\ Mgh\gamma_1,\ \boldsymbol{N}_{z'} = 0.$$

The equations of motion become considerably simplified owing to the fact that on account of the symmetry of the top $P = Q$. To integrate them completely we require three mutually independent relationships, and as such we choose the simplest, namely the third of the equations (505), which when integrated gives :

$$(\gamma_1 p + \gamma_2 q)P + \gamma_3 rR = \text{const.} \quad . \quad . \quad (533)$$

secondly, the third of the equations (506), which gives :

$$r = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (534)$$

and, lastly, the principle of vis viva, which here runs, by (510) and (357) :

$$\frac{1}{2}(Pp^2 + Qq^2 + Rr^2) + Mgz_0 = \text{const.}$$

or, by (532) and (534) :

$$P(p^2 + q^2) + 2Mgh\gamma_3 = \text{const.} \quad . \quad . \quad (535)$$

To these must be added the general relationships which link up the quantities p, q, r and γ_1, γ_2, γ_3.

Suppose that in the initial state the top rotates only about its axis of symmetry; that is, for $t = 0$, let $p = 0$, $q = 0$, $r = r_0$, and let the axis of symmetry make an angle θ_0 with the vertical, acute or obtuse, according as the centre of gravity S lies above or below the point of rotation O. Then the three equations of motion are :

$$(\gamma_1 p + \gamma_2 q)P + \cos\theta \cdot r_0 R = \cos\theta \cdot r_0 R$$

$$r = r_0$$

$$P(p^2 + q^2) + 2Mgh\cos\theta = 2Mgh\cos\theta_0.$$

We now reduce all the variables to terms of the independent angles θ, ϕ, ψ, by replacing γ_1 and γ_2 by (528), and p, q, r by (504) and (531). It then follows that :

$$P \sin^2 \theta \dot{\phi} = R r_0 (\cos \theta_0 - \cos \theta) \quad . \quad (536)$$

$$\cos \theta \cdot \dot{\phi} + \dot{\psi} = r_0 \quad . \quad . \quad . \quad (537)$$

$$P(\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) = 2Mgh(\cos \theta_0 - \cos \theta) \quad . \quad (538)$$

The first and the third equations enable us to calculate θ and ϕ, and then we can calculate ψ from the second equation. The elimination of $\dot{\phi}$ from (536) and (538) gives :

$$\dot{\theta}^2 = \frac{\cos \theta_0 - \cos \theta}{P} \cdot \left(2Mgh - \frac{R^2 r_0^2 (\cos \theta_0 - \cos \theta)}{P \sin^2 \theta}\right). \quad (539)$$

and from this we obtain t as an elliptic integral in θ.

We shall carry the calculation further for the case, which is interesting physically, where the velocity of rotation r_0 is very great, or, more accurately expressed, where :

$$r_0 >> \frac{MPgh}{R^2} \quad . \quad . \quad . \quad . \quad . \quad (540)$$

since a relationship between magnitudes has a physical meaning only when it is independent of the choice of the units of measure.

If we now set :

$$\theta = \theta_0 + \theta' \quad . \quad . \quad . \quad . \quad . \quad (541)$$

it follows at once that θ' is positive. For, by (538), $\theta_0 < \theta$; that is, the axis of symmetry of the top is steepest at the beginning.

Hence on the right-hand side of (539) the second factor is also positive; that is :

$$\frac{R^2 r_0^2 (\cos \theta_0 - \cos \theta)}{P \sin^2 \theta} < 2Mgh$$

or, in view of (540) :

$$\cos \theta_0 - \cos \theta << 1,$$

that is, θ' is small, and therefore, approximately :

$$\cos\theta_0 - \cos\theta = \theta' . \sin\theta_0.$$

Substituted in (539) this gives :

$$\left(\frac{d\theta'}{dt}\right)^2 = \frac{\theta' \sin\theta_0}{P}\left(2Mgh - \frac{R^2 r_0{}^2 \theta'}{P \sin\theta_0}\right)$$

which, when integrated, the initial conditions being taken into account, gives :

$$\theta' = \frac{2MPgh \sin\theta_0}{R^2 r_0{}^2} \sin^2\left(\frac{Rr_0 t}{2P}\right) \quad . \qquad (542)$$

Thus the angle of inclination θ of the axis of the top to the vertical fluctuates to and fro periodically between its smallest value θ_0 and a value very little different from it, the period being independent of the acceleration due to gravitation. The greater the velocity of rotation of the top, the more rapid and the smaller the fluctuations.

If we now fix our attention on the angle ϕ which the vertical plane that passes through the axis of the top makes with a vertical plane fixed in space, we get for it, from (536), (541) and (542) :

$$\frac{d\phi}{dt} = \frac{2Mgh}{Rr_0} \sin^2 \frac{Rr_0 t}{2P}.$$

Integration gives :

$$\phi = \frac{Mgh}{Rr_0}\left(t - \frac{P}{Rr_0} \sin\frac{Rr_0 t}{P}\right) + \phi_0 \quad . \quad . \qquad (543)$$

That is, the axis of the top performs a "precession," in which its vertical plane constantly rotates in a definite sense, with an angular velocity which varies periodically from a zero to a maximum value and which is independent of the angle of inclination of the axis of the top to the vertical. In this case, too, the fluctuations occur the more rapidly and are the smaller, the greater the velocity of rotation, whereas the mean angular velocity decreases as the number of rotations increases.

The sense of the precessional motion corresponds with the sign of r_0. Thus if $r_0 > 0$, the centre of gravity S on Fig. 43 moves away from the reader. We can characterize the relationship between the direction of the gravitational force, the position of the centre of gravity, the sense of the top's motion and the precessional motion in a manner independent of the choice of the directions of the axes by the simple theorem that in the precessional motion the positive direction of the axis of the top moves towards the positive direction of the turning moment due to the gravitational force. The latter passes from the front to the rear in Fig. 43.

This relationship of course persists if, in place of the force of gravity Mg, any other force F, say a blow, acts at any point S of the axis of the top.

A convenient way of visualizing the content of the last theorems is to observe an ordinary toy top, which has been set into rapid rotation and placed with its lower point on the floor so that the axis makes any arbitrary angle with the vertical.

For since the rate of rotation is gradually retarded through frictional resistances the different motions that correspond to the different values of r_0 gradually present themselves in succession. At first the axis appears to stand almost still, whereas actually it is performing very rapid fluctuations of small amplitude both in the direction of the vertical as well as perpendicularly to it. It then slowly begins to precess in the sense above indicated. At the same time its inclination to the vertical begins to fluctuate appreciably, first only a little and not very definitely, but then more and more violently, while at the same time the precessional motion becomes more and more rapid until finally the axis of the top leans over so far that a point on the circumference comes into contact with the floor.

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